On the Talmud division:

equity and robustness

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Abstract. The Talmud Division is a very old method of sharing developed by the rabbis in the Talmud and brought to the fore in the modern area some authors, among them are Aumann and Maschler. One compares the Talmud Division to other methods, mainly here the most popular, Aristotle's Proportional Division, but also to the equal division. The Talmud Division is more egalitarian than the Proportional Division for small levels of estate and conversely and it protects the weakest –those who cannot place a non-zero claim–. This suggests that claimants may choose among the claiming methods depending on their interest, what implies a metagame. Unlike other methods as the Proportional Division, the Talmud Division is not robust because the solution depends on the order in which groups of claimants are formed, while it could be impossible to form coalitions without following the increasing order of claimants or to find a general agreement about what precise coalition must be chosen. For a larger number of claimants, fulfilling the order-preserving condition may oblige to backtrack for a very large number of steps what implies an unreasonable volume of computations. The paper discusses also of three generalizations of the Contested Garment method to three or more claimants.

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1. Introduction

Historically, there are three main methods (or rules) to share a given quantity of commodity between two people: the equal division, which is obvious, Aristotle's Proportional Division (Rackham 1934; see comments in Kraut 2005) and the Talmud Division, that is, the "contested garment rule". Actually, many other methods are possible (Young 1987; Dagan et al. 1997; Moulin 2002, 2003; Thomson 2003; Balinski 2005) but they are not considered here. All methods are more than two thousand years old. The sharing may concern bankruptcy with each creditor being a claimant (as in Aumann and Maschler's paper), or it may concern actors being paid a percentage from admission charges to see a movie, or shareholders receiving corporate dividends, etc. The important idea is that claims may be unequal but they are given. For these examples, our contemporary reflex would be to make a sharing on a pro-rata basis or to make an equal sharing. But it is the great merit of some recent papers (O'Neill 1982; Aumann and Maschler 1985; Young 1987; Benoit 1997; Dagan et al. 1997; Herrero and Villar 2001; Aumann 2002; Moulin 2002, 2003; Thomson 2003; Hokari and Thomson 2003; Moreno-Ternero and Villar 2004; Dominguez and Thomson 2006)[1] is to show that the contested garment rule could be also chosen. Even if the equal division and the proportional division are better known and more common, the Talmud method is also interesting and is not just in an historical perspective. It allows a completely different sharing of a given estate between claimants. The key point of this paper consists into examining if the Talmud Division is a method that could be utilized in the real world for the many sharing problem that may occur or if it is only an interesting but historically anecdotic method.[2]

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[1] Dagan et al. (1997, p. 58) indicate that the method has also been studied by Pineles (1861), unfortunately for us in a text in Hebrew, not translated.

[2] This paper is no way critical of the Jewish Talmud or the Ketubot: no disrespect is intended to the very ancient Jewish religion. The paper discusses the merits and demerits of this fascinating method of division that can be deduced from the Talmud and the Ketubot, as exposed initially by O'Neill (1982) and Aumann and Maschler (1985). We know that two or three thousand years ago, mathematics was in its infancy and the authors and ancient commentators of the Talmud were in no position to make the deductions that we can do nowadays. Aumann and Maschler's paper is very instructive about the attempts at making a logical interpretation of the Talmud law of division.

One is able to compare the three methods by respect to equality and equity between claimants. Depending on the level of the total claimed, one method may be more favorable than the other for one claimant but less favorable for the other claimant. This opens the door for a game, which may be termed a *metagame* because it is a type of game played on the objectives. This is the subject of section 2. For three claimants or more, things are more complicated. The claimants are grouped into two groups –one alone and the other in one group– and the estate is shared between the two groups, then the group is subdivided into two, and so on. This poses the question of how the groups are made, in what order etc., remembering that the problem is combinatory, what poses the question of robustness. This is the subject of section 3. Section 4 is devoted to larger examples, with more than three claimants while section 5 discusses the possibility of generalizing the Contested garment method. Section 1 is this introduction. Section 6 concludes.

2. Equality, equity and metagame

2.1. Reminder

The traditional Talmudic writings consider the so-called contested garment problem: two persons claim for a garment, one the whole garment, the other half of the garment. The Talmud says that the first claimant receives what is not claimed by the other, that is, half of the estate, then the half remaining is shared equally, that is, the first claimant receives $3/4$ and the second $1/4$. Some explanations are required before going any farther. Consider two claimants, denoted 1 and 2, and E the total estate to be distributed; d_1 and d_2 are the amounts claimed by the individuals with $d_1 + d_2 \geq E$ (the commodity to be shared is scarce: Moulin (2003 p. 261) calls this "deficit-sharing" or "rationing"). The quantities x_1 and x_2 are what the claimants receive such that the estate is entirely shared out: $x_1 + x_2 = E$. Following the above authors, the Talmud (in the Seder *Nezikin*, tractate of *Metsi'a*, and in the *Ketubot*) consider the case where $d_i \leq E$ for all *i*. Each claimant receives what is not claimed by the other, that is, $E - d_2$ and $E - d_1$ respectively; what remains (that is, $d_1 + d_2 - E$) is shared equally between both individuals leading to the following allocations: $x_1 = \frac{1}{2}(E + d_1 - d_2)$ and $x_2 = \frac{1}{2}(E - d_1 + d_2)$. If a claimant claims more than what is possible *E*, that is, if $d_i > E$, the Talmud adds an important rule: his claim is replaced by *E*, that is, d_i is replaced by $\tilde{d}_i$ for all *i*:

(1) $\tilde{d}_i = \min(d_i, E)$ for all *i*

Moulin (2003, p. 37-38-262) calls this "truncation". It must be noted that a different but equivalent formula is possible: if $E \le \frac{1}{2}(d_1 + d_2)$, $x_i = \min\left(\lambda, \frac{d_i}{2}\right)$ for all *i* where λ is chosen by trial and error such that $x_1 + x_2 = E$; if $E > \frac{1}{2}(d_1 + d_2)$ one proceeds by symmetry.[3]

The Talmud Division seems complicated and unclear; hence the many discussions of the Jewish masters about it and the many scholarly papers. However, and this has never been brought to the fore,[4] the method corresponds to the minimization of the *Least Squares* –very popular among econometricians, even if no stochastic hypotheses are made here – (or of the quadratic mean) between vectors $\tilde{\mathbf{d}}$ and **x** under the constraint $\sum_{i=1}^{2} x_i = \sum_{i=1}^{2} \tilde{d}_i : \min_{x_i} \sum_{i=1}^{2} \left(x_i - \tilde{d}_i\right)^2$, s.t. $x_1 + x_2 = E$. The solution is $x_i = b + \tilde{d}_i$ for all $i = 1, 2$ where $b = \frac{1}{2}\left(E - \tilde{d}_1 - \tilde{d}_2\right)$. This criterion amounts to finding Gauss' orthogonal projection: the point of coordinates (x_1, x_2) is the orthogonal projection of the point $(\tilde{d}_1, \tilde{d}_2)$ on the plane $x_1 + x_2 = E$.[5]

The Figure 1, derived from Balinski's figures 1 and 2 (2005), reproduces the famous case of the contested garment. The line *BC* represents all the feasible allocations of the estate *E* between both individuals ($B(0, E)$ and $C(E, 0)$). Point *I* $\left(\frac{1}{2}E, E\right)$ shows what both agents claim. Point $T(E/4, 3E/4)$ is the allocation following the Talmud Division; point $I'(E/2, 5E/4)$, for which individual 1's claim exceeds the estate is also projected on *T*; $A(E/3, 2E/3)$ is the "Aristotle point".

Figure 1 about here

2.2. Equality and equity

It is interesting, but obvious, to compare the methods of division (namely, Talmud Division, Proportional Division and Constrained Equal Division) by respect to the criterion of equality: which method is the most egalitarian? By construction, the Constrained Equal

[3] Note that Diskin and Felsenthal (2007) argue that this solution can be considered as a modified Nash equilibrium.

[4] Young (1987, p. 410) indicates a completely different function, based on logarithms and exponentials.

[5] Remember that a distance and its square have the same minimum even if the square of a distance is not a distance.

Division is obviously the most egalitarian. Consider two individuals (or groups of individuals) with $d_2 > d_1$ (without loss of generality), the claims being given. It is elementary to say that when the estate is between zero and half of the total of the claims, that is, for $0 \leq E \leq \frac{1}{2}(d_1 + d_2)$, the Talmud Division is more egalitarian than the Proportional Division. When the estate is between half of the total of the claims and the total of the claims, that is, for $\frac{1}{2}(d_1 + d_2) \leq E \leq d_1 + d_2$, it is the Proportional Division which is more egalitarian than the Talmud Division.

This can be proved graphically (see Figure 2, derived from Balinski's figures 1, 2005). More precisely, the claims d_1 and d_2 are given but E is variable. With the Proportional Division, x_1 and x_2 are always such that $\frac{x_2}{x_1} = \frac{d_2}{d_1}$. When the estate is between 0 and d_1, the Talmud Division shares equally between each claimant; they receive the same quantity up to the estate of $\frac{1}{2}d_1$: the Talmud Division is more egalitarian than the Proportional Division. Between d_1 and $\frac{1}{2}(d_1 + d_2)$: only claimant 2 increases its allocation with the Talmud Division and catches up with 1, up to $E = \frac{1}{2}(d_1 + d_2)$ where the Proportional Division and the Talmud Division give the same result. Between $\frac{1}{2}(d_1 + d_2)$ and d_2, it is again claimant 2 who receives all the additional allocation with the Talmud Division, what increases the inequality beyond the Proportional Division. Between d_2 and $d_1 + d_2$, with the Talmud Division, 1 is again served, the same quantity being added equally to each allocation: this reduces inequality and for $E = d_1 + d_2$, Proportional Division and Talmud Division yield the same result.

Knowing that the Proportional Division implies the continuation of inequalities and that the Equal Division may be considered as wronging the strongest claimant, one deduces of these obvious findings that the Talmud Division tends to correct the repartition to the benefit of the weakest claimant when the estate is low, that is, in case of shortage: this should be considered as satisfactory in a morale point of view.

Figure 2 about here

The Talmud Division is order preserving (what is called Ranking Axiom by Moulin (2002)), that is, it conserves the inequality that can be observed before (that is, in the claims) and after distribution: $d_i > d_j \Leftrightarrow x_i > x_j$ for all *i*, *j*. The Proportional Division is also known to be order preserving; it does not affect inequality because proportionality remains always. Moreover, it may be noticed that the Talmud Division does not allow sharing if the estate is unknown *ex ante* while this is not so with the Proportional Division (as the division is always proportional to the claims, the estate will be shared *ex post* in these proportions).

Where is equity? This will be discussed for two claimants in order to compare the Talmud Division to other methods, focusing on the most popular, the Proportional Division. As pointed out by Aumann and Maschler (1985, p. 199), the individuals' claims may or may not be valid (see a very subtle analysis in Moulin (2002)). In a case of bankruptcy, the claims are always valid (and hence, given) because they are financial claims generated by the debts of a debtor and the claimants are relatively passive regarding the amount of debts that remain to be paid when bankruptcy occurs: in this case, the claims are what they are and their amount cannot be discussed. However, in some other cases, the validity of the claims can be discussed. In this case, each one could want to take the whole: if one individual claims less than the total, it is because he is weak or uninformed, unable to make a claim freely. The situations where the claims are free, at least for one of the claimants, the other being weak or not, will be discussed now.

If both agents can freely make their claims (hence, neither is weak), a good strategy with the Talmud Division consists in claiming the whole estate E (demanding more serves no purpose); but if both individuals do this, each one receives $\frac{1}{2}E$. With the Proportional Division, a good strategy consists in claiming an infinite amount, but if both choose to do this, each claimant will receive the same as the other one, that is, $\frac{1}{2}E$ also: there is no difference between the two methods of division.

If only one agent *j* is free to claim, with the Proportional Division, whoever claims nothing always receives nothing, while whoever claims a lot receives a lot: the weak are not protected. By contrast, the Talmud Division seems to correct the imbalance between the individuals. Whoever makes an excessive claim receives less than the total available: he is compelled to give up something to the other individual; whoever claims nothing receives a little, what is equitable. This is close to the idea of securement (Moreno-Ternero and Villar, 2004): "… securement says that any agent holding a feasible claim (a claim not larger than the estate) will get at least one *n*th of her claim, where *n* is the number of agents involved"; see also Dominguez and Thomson (2006). However, this is true only if the other individual does not claim all the estate: if *j* demands everything while *i* claims nothing, *i* receives zero: in Figure 1, the points $(E,0)$ and $(0,E)$ are projected on themselves; this is the only situation where a claimant receives nothing with the Talmud Division.

All this opens the door for a metagame where claimants may choose their methods of division depending of their own interest.

2.3. Metagame

Scholars generally suppose that the method is given and that claimants share between them following this method: the choice of a method is not in question, only how much each claimant receives is examined. If we return to Figure 2, we can see that for $0 \leq E \leq \frac{1}{2}(d_1 + d_2)$, claimant 1 earns more if the Talmud Division is chosen while claimant 2 prefers the Proportional Division; by contrast for $\frac{1}{2}(d_1 + d_2) \leq E \leq d_1 + d_2$, claimant 1 prefers the Proportional Division but 2 prefers the Talmud Division. For example, when $E = d_1$, 1 claims $\frac{1}{2}d_1$ with the Talmud Division (point *A*) while 2 claims $\frac{2}{3}d_1$ with the Proportional Division (point *E*); the segment *EA* contains all the feasible allocations. When $E = \frac{1}{2}(d_1 + d_2)$, each claimant agrees to claim $\frac{1}{2}d_1$ and $\frac{1}{2}d_2$ respectively (point *B*). When $E = d_2$, 1 claims $\frac{2}{3}d_1$ with the Proportional Division (point *F*) but 2 claims more with the Talmud Division, that is, $d_2 - \frac{1}{2}d_1$ (point *C*): segment *CF* contains all feasible allocations. For $E = d_1 + d_2$, both agree to claim d_1 and d_2 respectively (point *G*). As the result changes with the method chosen, if each claimant chooses the method he prefers, no agreement is possible: the claimants fall into a type of bilateral monopoly, with a Nash bargaining eventually.

At this point, it is useful to recall the traditional theory of the bilateral monopoly. The seller has a commodity in hand, that has cost *c* to be produced and the buyer has an amount of money *v* in hand, while the exchange is possible if $v \geq c$; the maximum surplus is equal to $v - c$. All mid-way solutions in the interval $]c, v[$ are acceptable by both because all of these solutions increase the welfare of each; the Nash solution is $\frac{1}{2}(c + v)$ for each. The most important point is that each agent always has the possibility of choosing not to exchange, the surplus being equal to zero for each: the seller keeps on his commodity of value *c* and the buyer his amount of money *v*. The interval must be completed by the bounds *c* and *v* to become $[c, v]$ because none of these solutions decreases anyone's welfare (Pareto optimality).

However, for the problem of division, things are quite different. Consider the case $E = d_1$. From the point of view of claimant 1, $c = \frac{1}{3}d_1$ determined by the Proportional Division is the lower limit (the Proportional Division is the worse method for him: he cannot accept to receive less) while $v = \frac{1}{2}d_1$ determined by the Talmud Division is the upper limit of claimant 2 (he cannot accept to give more). The point of view of claimant 2 is obviously equivalent: he cannot accept to receive less than $\frac{1}{2}x_1$ while claimant 1 will not accept to give him more than $\frac{2}{3}x_1$. All told, $\frac{1}{6}d_1$ is under discussion. If both are informed about all values (meaning they are able to calculate what the solutions of the Talmud Division and the Proportional Division could be), there is no rule to determine what the solution actually is: it is a trial of strength. In the bilateral monopoly, one proposes usually the Nash solution which leads here to share $\frac{1}{6}d_1$ equally between both, that is, to allocate $\frac{5}{12}d_1$ to claimant 1 and $\frac{7}{12}d_1$ to claimant 2: it amounts to using a method of division which is the average between the Talmud Division and the Proportional Division.

However, there is a big difference with the bilateral monopoly: in the problem of division, firstly, nobody has anything in his hand before dividing and secondly, choosing any method of division, even the least advantageous, will increase the welfare of both. Hence, the claimants must share anyway, contrarily to the bilateral monopoly, but if they are ultimately unable to agree on a common method of division, there is under-optimality. For example, returning to Figure 2, when $E = d_1$, from the point of view of claimant 1, no division means that claimant 1 looses $\frac{1}{3}d_1$ and claimant 2 looses $\frac{1}{2}d_1$ relative to their least advantageous method of division. If all this is generalized to any method of division, that is, to any continuous monotonic path between 0 and point *G* in Figure 2, *c* and *v* become equal to 0 and 1 respectively.

All these analyses conducted for two claimants concerning equality, inequality, equity and metagame seem rather elementary and thinking about more claimants could seem more satisfactory but we will see that there is a large impossibility for three claimants or more, unfortunately.

3. Robustness in the Three-Wives problem

The Contested garment case concerns two claimants: a method of division limited to only two claimants is just a toy. A real method must be able to handle many claimants: the Talmud method concerns also three claimants (the Three-Wives problem) but for three claimants or more, things are more complicated.

The Talmud uses a definite order of coalitions. Aumann and Maschler (1985) have made explicit the Talmud Division for more than two claimants: the coalitions that are not order-preserving are rejected; they term the new procedure the *coalitional procedure*. Aumann and Maschler (1985) give a formalization of this procedure termed as the *orderly step-by-step process* although I would prefer *orderly sequential coalitions*. After ordering the claims in increasing order, that is, $0 \leq d_1 \leq d_2 \leq d_3 \leq ...$, the first claimant forms a first group alone and the other claimants a second group: the division is done between both groups; then the second claimant forms a group alone while the 3rd up to the nth claimants form a second group, the division being done between both groups as before; etc.

The question is: how individuals are grouped in the Talmud Division, while for the Proportional Division, this question obviously does not matter. As far I know, never Aumann and Maschler or any followers have developed the idea, at least in a published paper but they suggest that some further studies must be conducted.[6] Here, we want to examine the question. We define *robustness* as the following property: a method of division is robust if the shares that solve the problem do not depend on how claimants are grouped and in what order they are grouped.[7]

The Talmud examines the case of a man who has three wives; the contracts of marriage say that each woman should receive 100, 200 and 300 at its death. The man dies but its estate is only 100: each wife receives $100/3$. The Talmud says also that if the estate would have been equal to 200, they should receive 50, 75 and 75 while if it would be 300, they should receive 50, 100 and 150. Aumann et Maschler (1985) show that this result is found by grouping first wives 2 et 3 together and by sharing between this group and wife 1 then by sharing between the wives 2 and 3. Hence, a particular order is chosen, the coalitional procedure.

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[6] The following quotation shows that Aumann and Maschler new the importance of the order of formation of coalitions, and that the results depend of the particular choice of starting from the poorest claimant (Aumann and Maschler, 1987, p. 207):

"Theorem C applies the CG principle [i.e., Talmud Division] to pairs of coalitions; and it describes an orderly step-by-step process (the rule of sequential coalitions), which by its very definition must lead to a unique result. But it uses only certain carefully selected pairs of coalitions, not all such pairs. As described –with the coalitions $\{1\}$ and $\{2,...,n\}$– the coalitional procedure yields a monotonic rule and order-preserving solutions. Moreover, it appears that they are the only such coalitions, though we have no satisfactory formulation and proof of such a result. Also, it appears that if the creditors may form coalitions as they wish, then for $E \leq D/2$, the incentives lead to the coalition suggested by the coalitional procedure. These matters call for further study".

[7] If the groups are coalitions, the idea of robustness is closed the idea of coalition-proof in the sense of Bernheim *et al*. (1987).

3.1. Three different coalitions

The choice of a particular order in the sequential procedure of coalitions is obviously arbitrary because some other ways of grouping are possible: the claimants can choose to form various coalitions as they wish (e.g. 1 and 2 versus 3, or 1 and 3 versus 2, or 2 and 3 versus 1), what changes the result and is not always order-preserving.

Coalition {2,3} against 1 (the Talmudic problem of the Three Wives)

When it is applied to the estate division problem (wives 1, 2 and 3 claiming respectively 100, 200 and 300), wives 2 and 3 form a group versus wife 1, then wives 2 and 3 share between them. The allocations must be order preserving for awards and losses; if not the process is stopped and what remains is affected equally between the remaining claimants. It is for example the case if the estate is equal to 100 (the process is stopped at the first step because the orderly step-by-step process would give 50 for claimant 1 and only 50 for the two other, that is, $(50, 25, 25)$, what is not order preserving for the awards). It is also the case when the estate is equal to 500 (the process is also stopped at the first step: the orderly step-by-step process would lead to a loss of 50 for the first claimant and of 50 for the two others, that is, an allocation of $(50, 175, 275)$). We call this the *special equality rule*. The special equality rule –an equal allocation between all remaining claimants– is not arbitrary: it corresponds to the minimum award (or loss) that can be affected to the claimants without violating the order. For example, if the estate is 100, claimant 1 must receive $33\frac{1}{3}$ to the maximum in order to let $66\frac{2}{3}$ to the two others; claimant 2 itself must receive $33\frac{1}{3}$ to the maximum in order to let $33\frac{1}{3}$ to claimant 3. Actually, after giving its maximum to claimant 1, the rule follows the Talmud Division for the other claimants: after giving $33\frac{1}{3}$ to 1, it comes $33\frac{1}{3}$ to 2 and $33\frac{1}{3}$ to 3 by Talmud Division between 2 and 3. In Table 1, we have computed many cases between zero and the value of the estate: they will be useful later when we will examine some problems.

Claimants and rights / Estates	Awards			Losses		
	1: 100	2: 200	3: 300	1: 100	2: 200	3: 300
0	0	0	0	100	200	300
100	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$	$66\frac{2}{3}$	$166\frac{2}{3}$	$266\frac{2}{3}$
150	50	50	50	50	150	250
200	50	75	75	50	125	225
225	50	87.5	87.5	50	112.5	212.5
250	50	100	100	50	100	200
300	50	100	150	50	100	150
350	50	100	200	50	100	100
375	50	112.5	212.5	50	87.5	87.5
400	50	125	225	50	75	75
450	50	150	250	50	50	50
500	$66\frac{2}{3}$	$166\frac{2}{3}$	$266\frac{2}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
600	100	200	300	0	0	0

Table 1. The Talmudic marriage problem: 2 and 3 form a coalition

One observes that this table is symmetric by respect to $E = D/2 = 300$, as expected; it is also monotonic by respect to E and order-preserving (after applying the special equality rule for the cases $E = 100$ and $E = 500$; remark that $E = 150$ and $E = 450$ do not need to apply the special equality rule).

Table 1 applies when Aumann-Maschler's orderly step-by-step procedure is applied: the lower claimant is left alone, the two other for a group and the estate is shared between claimant 1 and the group 2-3 following the Talmud Division; then the estate is shared between claimants 2 and 3 following the Talmud Division.

Other coalition: {1,2} against 3

We have computed the table when the coalition is made between claimants 1 and 2 instead of 2 and 3 in Table 2.

Claimants and rights / Estates	Awards			Losses		
	1: 100	2: 200	3: 300	1: 100	2: 200	3: 300
0	0	0	0	100	200	300
100	25	25	50	75	175	250
150	37.5	37.5	75	62.5	162.5	225
200	50	50	100	50	150	200
225	50	62.5	112.5	50	137.5	187.5
250	50	75	125	50	125	175
300	50	100	150	50	100	150
350	50	125	175	50	75	125
375	50	137.5	187.5	50	62.5	112.5
400	50	150	200	50	50	100
450	62.5	162.5	225	37.5	37.5	75
500	75	175	250	25	25	50
600	100	200	300	0	0	0

Table 2. The Talmudic marriage problem with coalition 1-2 against 3

This table is again symmetric by respect to $E = D/2 = 300$, monotonic by respect to E and order preserving. One remarks that in no case applying the special equality rule is necessary. Coalition 2-3 is not the only coalition that fulfils the three properties (symmetry, monotony, order-preserving) at the same time contrarily to what is said by Aumann and Maschler (1985, p. 207). Notice that claimant 1 receives only 25 when $E = 100$ what is under $\frac{1}{3}$ of his claim: the rule of "securement"[8] (Moreno-Ternero and Villar 2004; Dominguez and Thomson 2006) is violated with the coalition 1-2 against 3.

Coalition: {1,3} against 2

If we consider the case of the coalitions which violate the order of claims, as 1-3 against 2, we are obliged to apply the rule of order preservation. We obtain what is indicated in Table 3. A difficulty must be noted: monotony may be violated by the special equality rule: see the discussion in Annex 1. Moreover, even the special equality rule may be incoherent in some circumstances: if $225 < E < 250$ the Talmud division is not monotonic. The result is not Talmudic and completely conditioned by the monotony axiom and there is a clear discontinuity for $E = 250$. See the discussion in Annex 2.

Claimants and rights / Estates	Awards			Losses		
	1: 100	2: 200	3: 300	1: 100	2: 200	3: 300
0	0	0	0	100	200	300
100	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$	$66\frac{2}{3}$	$166\frac{2}{3}$	$266\frac{2}{3}$
150	50	50	50	50	150	250
200	50	$66\frac{2}{3}$	$83\frac{1}{3}$	50	$133\frac{1}{3}$	$216\frac{2}{3}$
225	50	75	100	50	125	200
250	50	100	100	50	100	200
300	50	100	150	50	100	150
350	50	100	200	50	100	100
375	50	125	200	50	75	100
400	50	$133\frac{1}{3}$	$216\frac{2}{3}$	50	$66\frac{2}{3}$	$83\frac{1}{3}$
450	50	150	250	50	50	50
500	$66\frac{2}{3}$	$166\frac{2}{3}$	$266\frac{2}{3}$	$33\frac{1}{3}$	$133\frac{1}{3}$	$233\frac{1}{3}$
600	100	200	300	0	0	0

Table 3. The Talmudic marriage problem with coalition 1-3 against 2

[8] This excerpt from (Moreno-Ternero and Villar, 2004): "… securement says that any agent holding a feasible claim (a claim not larger than the estate) will get at least one *n*th of her claim, where *n* is the number of agents involved". See also Dominguez and Thomson (2006).

3.2. Dominance between coalitions

Table 4 summarizes the awards for the three tables above. One remarks that for $E = 0$, $E = D/2$ and $E = D$, that is, for $E = 0$, $E = 300$ and $E = 600$, the result does not depend on the coalitions.

Estates \ Coalitions	Claimant 1			Claimant 2			Claimant 3		
	{2,3} (alone)	{1,2}	{1,3}	{2,3}	{1,2}	{1,3} (alone)	{2,3}	{1,2} (alone)	{1,3}
0	0	0	0	0	0	0	0	0	0
100	$33\frac{1}{3}$	25	$33\frac{1}{3}$	$33\frac{1}{3}$	25	$33\frac{1}{3}$	$33\frac{1}{3}$	50	$33\frac{1}{3}$
150	50	37.5	50	50	37.5	50	50	75	50
200	50	50	50	75	50	$66\frac{2}{3}$	75	100	$83\frac{1}{3}$
225	50	50	50	87.5	62.5	75	87.5	112.5	100
240	50	50	50	95	70	90	95	120	100
250	50	50	50	100	75	100	100	125	100
300	50	50	50	100	100	100	150	150	150
350	50	50	50	100	125	100	200	175	200
360	50	50	50	105	130	110	205	180	200
375	50	50	50	112.5	137.5	125	212.5	187.5	200
400	50	50	50	125	150	$133\frac{1}{3}$	225	200	$216\frac{2}{3}$
450	50	62.5	50	150	162.5	150	250	225	250
500	$66\frac{2}{3}$	75	$66\frac{2}{3}$	$166\frac{2}{3}$	175	$166\frac{2}{3}$	$266\frac{2}{3}$	250	$266\frac{2}{3}$
600	100	100	100	200	200	200	300	300	300
Claims	100			200			300		

Table 4. The Talmudic marriage problem: summary

A coalition *i* dominates a coalition *j* for a given claimant if the awards given by *i* are never lower than the awards given by *j* (but may be equal sometimes). We will say that any claimant prefers choosing a dominating coalition, although it can be equivalent to another for particular levels of estate. Fortunately, the problem exposed above for coalition 1-3 and its non-Talmudic solution when $225 < E < 250$ or $350 < E < 375$ does not affect the order of dominances whatever the choice made (see for example the cases of $E = 240$ and $E = 360$). If the estate is low, that is, $E \leq D/2$, we see that for claimant 1, $\{2,3\} \approx \{1,3\} \succ \{1,2\}$; for claimant 2, $\{2,3\} \succ \{1,3\} \succ \{1,2\}$; for claimant 3, $\{1,2\} \succ \{1,3\} \succ \{2,3\}$. We can say that claimant 1 never prefers being in a coalition with claimant 2 but is indifferent between the two other cases, namely being alone and being with claimant 3; claimant 2 prefers being with claimant 3, then being alone and finally being with 1; claimant 3 prefers being alone then being with 1, then being with 2. This can be summarized –always for lower estates– much more compactly in the following way, by showing the order of the preferences (the numbers are those of the claimants; 0 corresponds to the case where a claimant prefers to be alone):

Claimant 1: $0 \approx 3 \succ 2$

Claimant 2: $3 \succ 0 \succ 1$

Claimant 3: $0 \succ 1 \succ 2$

If we assume that the claimants may choose themselves their coalitions, we deduce that there is an obvious impossibility. Claimant 3 prefers to be alone: we may assume that he remains alone but this obliges claimants 1 and 2 to form a coalition, what is impossible because they dislike being together. If claimant 2 chooses its first best, being with 3, claimant 1 accepts to be alone (its first best equally placed) but claimant 3 is forced to choose its worse choice. If claimant 1 chooses also its first best, he can be alone but the case has been explored as being the worse for 3 or he can join 3 but this is the second best of 3 what forces 2 to choose also its second best. One observes also that the coalition $\{2,3\}$, 1 remaining alone, –the case considered by the Talmud– is not realistic because claimant 3 completely dislikes being with 2 even if being alone is one of the best choice of claimant1. This refutes Aumann and Maschler's assertion (1985, p. 207) that the incentives lead to coalition $\{2,3\}$.

The order of the coalitions, that is, of their "probabilities" based on claimants' preferences, is the following: $\{1,3\} \approx \{2,3\} \succ \{1,2\}$: coalition $\{1,3\}$ violates one level of preferences of two claimants (2 and 3), coalition $\{2,3\}$ violates two levels of preferences of one claimant (claimant 1) while coalition $\{1,2\}$ violates two levels of preferences of two claimants (claimants 1 and 2): unexpectedly, coalition $\{1,3\}$ that violates the order of claimants is among the most probable, along with coalition $\{2,3\}$ considered by the Talmud; claimant 3, the richest, is present in the two most probable cases.

Everything is obviously reversed for large estates, that is, $E \geq D/2$:

Claimant 1: $2 \succ 0 \approx 3$

Claimant 2: $1 \succ 0 \succ 3$

Claimant 3: $2 \succ 1 \succ 0$

and the "probabilities" of preferences becomes: $\{1,2\} \approx \{2,3\} \succ \{1,3\}$.

3.3.The Three-Wives problem with other levels of rights

In the Talmudic Three-Wives problem, claimant 2's claim is at equal distance of the two other claims (actually, it is the mean of the two other claims). Hence, we have also completely tested the case where (i) claimant 2's claim is close to those of claimant 1, with the claims $\mathbf{d} = (100, 110, 390)$ and D unchanged at 600 and (ii) claimant 2's claim is close to those of claimant 3, with the claims $\mathbf{d} = (100, 240, 260)$ and $D = 600$.

$\mathbf{d} = (100, 110, 390)$ and $D = 600$: see Table 5.

Estates \ Coalitions	Claimant 1			Claimant 2			Claimant 3		
	2-3 (alone)	1-2	1-3	2-3	1-2	1-3 (alone)	2-3	1-2 (alone)	1-3
0	0	0	0	0	0	0	0	0	0
100	$33\frac{1}{3}$	25	$33\frac{1}{3}$	$33\frac{1}{3}$	25	$33\frac{1}{3}$	$33\frac{1}{3}$	50	$33\frac{1}{3}$
150	50	37.5	50	50	37.5	50	50	75	50
160	50	40	50	55	40	55	55	80	55
200	50	50	50	55	50	55	95	100	95
225	50	50	50	55	55	55	120	120	120
240	50	50	50	55	55	55	135	135	135
250	50	50	50	55	55	55	145	145	145
300	50	50	50	55	55	55	195	195	195
350	50	50	50	55	55	55	245	245	245
360	50	50	50	55	55	55	255	255	255
375	50	50	50	55	55	55	270	270	270
400	50	50	50	55	60	55	295	290	295
440	50	60	50	55	70	55	335	310	335
450	50	62.5	50	60	72.5	60	340	315	340
500	$66\frac{2}{3}$	75	$66\frac{2}{3}$	$76\frac{2}{3}$	85	$76\frac{2}{3}$	$356\frac{2}{3}$	340	$356\frac{2}{3}$
600	100	100	100	110	110	110	390	390	390
Claims	100			110			390		

Table 5. The Talmudic marriage problem: summary for 2's claim close to 1's claim

Monotony is violated for $E \to 160^-$: $\lim_{E\to 160^-} \mathbf{x} = \left(50, 53\frac{1}{3}, 56\frac{2}{3}\right)$ but $\mathbf{x} = (50, 55, 55)$ if $E = 160$ (by symmetry, it is the same thing for $E \to 440^+$). The solution is conditioned by monotony.

In Table 5, if the estate is low, that is, $E \le D/2$, we see that for claimant 1, $\{2,3\} \approx \{1,3\} \succ \{1,2\}$; for claimant 2, $\{2,3\} \approx \{1,3\} \succ \{1,2\}$; for claimant 3, $\{1,2\} \succ \{1,3\} \approx \{2,3\}$. Hence:

Claimant 1: $0 \approx 3 \succ 2$

Claimant 2: $3 \approx 0 \succ 1$

Claimant 3: $0 \succ 1 \approx 2$

Now, being alone is among the best choices of all claimants, even if they must form a coalition. Unfortunately, the other best choice of claimants 1 and 2 is claimant 3: if one of them wants to form a coalition with claimant 3, this choice violates the preferences of claimant 3! And if claimant 3 has satisfaction by remaining alone, it is the worst choice of claimants 1 and 2. Things are reversed if $E \ge D/2$.

$\mathbf{d} = (100, 240, 260)$ and $D = 600$: see Table 6.

Again we experience difficulties with the special equality rule for $150 < E < 290$ in coalition $\{1,2\}$: see Annex 3. The same problem occurs with coalition $\{1,3\}$. These difficulties lead to consider intervals for the value of allocations. The results are summarized in Table 6.

	Claimant 1			Claimant 2			Claimant 3		
Coalitions / Estates	2-3 (alone)	1-2	1-3	2-3	1-2	1-3 (alone)	2-3	1-2 (alone)	1-3
0	0	0	0	0	0	0	0	0	0
100	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$	50	$33\frac{1}{3}$
150	50	50	50	50	50	50	50	50	50
200	50	50	50	75	65-75	$66\frac{2}{3}$	75	85-75	$83\frac{1}{3}$
225	50	50	50	87.5	65-87.5	77.5-87.5	87.5	85-87.5	97.5-87.5
250	50	50	50	100	90-100	90-100	100	110-100	110-100
290	50	50	50	120	110	120	120	130	120
300	50	50	50	120	120	120	130	130	130
310	50	50	50	120	130	120	140	130	140
350	50	50	50	140	150-140	150-140	160	150-160	150-160
375	50	50	50	152.5	175-152.5	152.5-162.5	172.5	175-172.5	162.5-172.5
400	50	50	50	165	175-165	$173\frac{1}{3}$	185	175-185	$176\frac{2}{3}$
450	50	50	50	190	190	190	210	210	210
500	$66\frac{2}{3}$	$66\frac{2}{3}$	$66\frac{2}{3}$	$206\frac{2}{3}$	190	$206\frac{2}{3}$	$266\frac{2}{3}$	185	$226\frac{2}{3}$
600	100	100	100	240	240	240	260	260	260
Claims	100			240			260		

Table 6. The Talmudic marriage problem: summary for 2's claim close to 3's claim

In this Table 6, if the estate is low, that is, $E \leq D/2$, we see that for claimant 1, $\{2,3\} \approx \{1,3\} \approx \{1,2\}$. Ranking claimants 2 and 3 cannot be done rigorously. For claimant 2, $\{2,3\}$ is the preferred but it is impossible to say if $\{1,3\}$ is better than $\{1,2\}$; for claimant 3, $\{2,3\}$ is dominated but $\{1,2\}$ and $\{1,3\}$ are not clearly separated. Hence we could write not rigorously:

Claimant 1: $0 \approx 3 \approx 2$

Claimant 2: $3 \succ 0 \approx 1$

Claimant 3: $0 \approx 1 \succ 2$

The only clear thing is that 1 is completely indifferent while 3 does not want to be with 2 but 2 prefers to be with 3. Things are reversed if $E \geq D/2$.

4. Larger examples

Returning to Aumann-Maschler's larger examples is also interesting. In their footnote 23, Aumann and Maschler (1985, p. 207) assume that for their example (2), the solution $\mathbf{x} = (50\ 100\ 120\ 120\ 120)$ is the only one which is order-preserving but they declare to have no demonstration. We have found at least one other set of coalitions which is order preserving on awards and losses: $(\{1,2,3\},\{4,5\})$, $(1,\{2,3\})$, $\{2,3\}$ and $\{4,5\}$ giving $\mathbf{x}=(50\ 100\ 105\ 127.5\ 127.5)$; it can be verified that these sets of coalitions are monotonic and order-preserving for awards and losses for $300 \le E \le 1200$. This set violates the implicit rule that places one claimant alone at each step but if the claimants are free to form coalitions, they must be able to group themselves as they wish.

Moreover, if we work in reverse order from claimant 5 to claimant 1, that is, if we form the set of coalitions $(5,\{4,3,2,1\})$, $(4,\{3,2,1\})$ then $(3,\{2,1\})$ and $\{2,1\}$, we find the new solution $\mathbf{x}=(32.5\ 32.5\ 65\ 130\ 250)$. This one is also order-preserving on awards and losses but different from the previous solution. One immediately notices that the 5th claimant, the strongest, receives more than when he was in a coalition: it might be more advantageous not to join coalitions! This is probably paradoxical.

One question arises. Why would a particular claimant reject a division advantageous for him only because it does not preserve the order of the claims? The only explanation is: because the other claimants would also give up the coalition if they are losing in it. This is the case in the above example (2): claimant 3 could receive 150 if claimants 4 and 5 form a coalition but claimants 4 and 5 have no incentive to do this as they would lose at least 15 each by so doing; for them, it is not a matter of preserving the order or not, but a matter of not losing out.

Obtaining results that are dependent on the way the claimants are grouped is an obvious weakness of the Talmud Division: in a word, the Talmud Division is not robust while the Proportional Division is obviously known to be robust, as all global consistent methods, – "global" meaning that, for a division problem among *n* individuals, the result can be found by maximizing or minimizing only one time a function without any iterative or step-by-step process (which can be parametric or objective: see Young 1987)–. In the context of Jewish law (the *Mishna*), the given order of coalitions (from the larger claim to the lower, this one being always alone) may perfectly be taken as a fact, hence the lack of robustness but a method of division that is not always order-preserving cannot be considered as fair because it is to be expected that a fair method should yield a unique result regardless of how it is applied. Remark that the above paradoxes are not contradictory with consistency: once the order of grouping is chosen, the method is consistent, but the result varies with this order.

Larger examples pose the problem of backtracking. Consider the following example (Aumann and Maschler 1985, p. 207):

(2) $\mathbf{d}=(100\ 200\ 300\ 400\ 500)$ and $E=510$

The solution of the Talmud Division is $\mathbf{x} = (50\ 100\ 120\ 120\ 120)$, the coalitions being $(1, \{2,3,4,5\})$ and then it is $(2, \{3,4,5\})$; after that, equality is applied between 3, 4 and 5 (yielding 120 to each) because no other coalitions after $(2, \{3,4,5\})$ would be order preserving: Aumann and Maschler quote that if the coalition $(3, \{4,5\})$ is formed, it comes $x_3 = 150$: 4 and 5 will receive less than 105.

Stopping the non-order-preserving process could force going backtrack on the tree structure for two or more coalitions (and not only for one coalition). A simple example will demonstrate it. Consider six claimants, with $\mathbf{d} = (100\ 200\ 300\ 400\ 500\ 600)$ and $E = 610$. The first coalition $(1, \{2,3,4,5,6\})$ leads to $x_1 = 50$, the second coalition $(2, \{3,4,5,6\})$ to $x_2 = 100$, the third coalition $(3, \{4,5,6\})$ to $x_3 = 150$, the fourth $(4, \{5,6\})$ to $x_4 = 155$ and finally $x_5 = x_6 = 77.5$: the order is not preserved. Hence claimants 5 and 6 do not accept to enter in the fourth coalition, so they backtrack; now equality give claimants 4, 5 and 6 only 103.3 each, less than for claimant 3; again, claimants 4, 5 and 6 backtrack and do not join the third coalition, which gives 115 to claimants 3, 4, 5 and 6 each. Finally, the result is $\mathbf{x} = (50\ 100\ 115\ 115\ 115\ 115)$. This mechanism supposes that the players may play "for fun", that is, they are able to compute all the results of the future coalitions. Here, they have to compute for two coalitions only, but for a million people, they would have to compute for an unreasonable number of coalitions.

Notice that Moulin (2003, p. 58) quote another problem from the Talmud, the following inheritance problem. Four brothers, Reuben, Simeon, Levi and Judah claim respectively $d_R = 1$, $d_S = 1/2$, $d_L = 1/3$ and $d_J = 1/6$ of their father's estate (hence, $E = 1$). The solution is obviously $x_i = \frac{1}{2} d_i$, that is, $x_R = 1/2$, $x_S = 1/4$, $x_L = 1/6$ and $x_J = 1/12$. It does not depend on the coalitions formed and it corresponds also to the Shapley value. This is because $E = D/2$: we insist to say that it is a very special case.

5. Generalizing the Contested Garment

If the sequential coalitions are not a good generalization of the Contested Garment, that is, the Talmud Division for two claimants, one may wonder if it is possible to find another method, remembering that any generalization must allow retrieving the results of the two-claimants case.

The "run to the bank", that is, the Shapley value, mentioned by Moulin (2003, pp. 57-58), does not gives the same result than any of the three coalitions (except for $E = 0$,

$E = D/2$ and $E = D$, that is, for $E = 0$, $E = 300$ and $E = 600$) as indicated in Table 7. Moulin pedagogically explains that in the run to the bank, the claimants arrive randomly, those arrived first takes what he claims (or the totality if what is available is lower than its claim) then if it remains something, those arrived second does the same, etc. Assuming that these cases are equally probable, the expectation is the searched result, which corresponds to the Shapley value.

Claimants and rights / Estates	Awards			Losses		
	1: 100	2: 200	3: 300	1: 100	2: 200	3: 300
0	0	0	0	100	200	300
100	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$	$66\frac{2}{3}$	$166\frac{2}{3}$	$266\frac{2}{3}$
150	$33\frac{1}{3}$	$58\frac{1}{3}$	$58\frac{1}{3}$	$66\frac{2}{3}$	$141\frac{2}{3}$	$241\frac{2}{3}$
200	$33\frac{1}{3}$	$83\frac{1}{3}$	$83\frac{1}{3}$	$66\frac{2}{3}$	$116\frac{2}{3}$	$216\frac{2}{3}$
225	37.5	87.5	100	62.5	112.5	200
250	$41\frac{2}{3}$	$91\frac{2}{3}$	$116\frac{2}{3}$	$58\frac{1}{3}$	$108\frac{1}{3}$	$183\frac{1}{3}$
300	50	100	150	50	100	150
350	$58\frac{1}{3}$	$108\frac{1}{3}$	$183\frac{1}{3}$	$41\frac{2}{3}$	$91\frac{2}{3}$	$116\frac{2}{3}$
375	62.5	112.5	200	37.5	87.5	100
400	$66\frac{2}{3}$	$116\frac{2}{3}$	$216\frac{2}{3}$	$33\frac{1}{3}$	$83\frac{1}{3}$	$83\frac{1}{3}$
450	$66\frac{2}{3}$	$141\frac{2}{3}$	$241\frac{2}{3}$	$33\frac{1}{3}$	$58\frac{1}{3}$	$58\frac{1}{3}$
500	$66\frac{2}{3}$	$166\frac{2}{3}$	$266\frac{2}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
600	100	200	300	0	0	0

Table 7. The Talmudic marriage: the Shapley-value solution

The Least Squares, after applying truncation by formula (1), are not a generalization of the Talmud Division also: they do not give the same results, as showed by Table 8. One notices that the table is not symmetric and overall not monotonous, even if it is order preserving. Annoying for a method of division.

Claimants and rights / Estates	Awards			Losses		
	1: 100	2: 200	3: 300	1: 100	2: 200	3: 300
0	0	0	0	100	200	300
100	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$	$66\frac{2}{3}$	$166\frac{2}{3}$	$266\frac{2}{3}$
150	$16\frac{2}{3}$	$66\frac{2}{3}$	$66\frac{2}{3}$	$83\frac{1}{3}$	$133\frac{1}{3}$	$233\frac{1}{3}$
200	0	100	100	100	100	200
225	0	100	125	100	100	175
250	0	100	150	100	100	150
300	0	100	200	100	100	100
350	$16\frac{2}{3}$	$116\frac{2}{3}$	$216\frac{2}{3}$	$83\frac{1}{3}$	$83\frac{1}{3}$	$83\frac{1}{3}$
375	25	125	225	75	75	75
400	$33\frac{1}{3}$	$133\frac{1}{3}$	$233\frac{1}{3}$	$66\frac{2}{3}$	$66\frac{2}{3}$	$66\frac{2}{3}$
450	50	150	250	50	50	50
500	$66\frac{2}{3}$	$166\frac{2}{3}$	$266\frac{2}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
600	100	200	300	0	0	0

Table 8. The Talmudic marriage: the Least Squares solution

In the Contested Garment", the Talmud Division between two claimants, those who claims the whole takes what is not claimed by the other –the half of the garment– and what remains is shared equally. This method, which can be called the method "You don't want it? I take it!", does not give the same results for any coalition when it is generalized, as showed by Table 9. This time, the table is symmetric, monotonous and order preserving.

Claimants and rights / Estates	Awards			Losses		
	1: 100	2: 200	3: 300	1: 100	2: 200	3: 300
0	0	0	0	100	200	300
100	0	50	50	100	150	250
150	0	50	100	100	150	200
200	$16\frac{2}{3}$	$66\frac{2}{3}$	$116\frac{2}{3}$	$83\frac{1}{3}$	$133\frac{1}{3}$	$183\frac{1}{3}$
225	25	75	125	75	75	175
250	$33\frac{1}{3}$	$83\frac{1}{3}$	$133\frac{1}{3}$	$66\frac{1}{3}$	$116\frac{1}{3}$	$166\frac{1}{3}$
300	50	100	150	50	100	150
350	$66\frac{1}{3}$	$116\frac{1}{3}$	$166\frac{1}{3}$	$33\frac{1}{3}$	$83\frac{1}{3}$	$133\frac{1}{3}$
375	75	125	175	25	75	125
400	$83\frac{1}{3}$	$133\frac{1}{3}$	$183\frac{1}{3}$	$16\frac{2}{3}$	$66\frac{2}{3}$	$116\frac{2}{3}$
450	100	150	200	0	50	100
500	100	150	250	0	50	50
600	100	200	300	0	0	0

Table 9. The Talmudic marriage: the "You don't want it? I take!" solution

6. Conclusion

We have compared the Talmud Division to other methods of division in terms of its tendency to maintain equality between claimants. The Talmud Division is more egalitarian than the Proportional Division for small levels of estate and conversely. The Talmud Division protects the weakest, those who cannot place a non-zero claim. This has led us to consider that claimants may want to choose the sharing methods depending on their interest, what implies that a metagame may be played: each claimant chooses the method which is more favorable for him. The Talmud Division is an attractive method of division when two claimants are considered and it may be used in the traditional cases as bankruptcy but also in those of actors being paid a percentage from admission charges to see a movie, of shareholders that receive corporate dividends or of many patrimonial problems as divorces.

When there are more than two claimants, the Talmud Division is applied sequentially. One may consider that coalitions are made freely what implies that the problem is combinatory. We have shown that the Talmud Division is not robust because the solution depends on the order in which groups are formed, unlike the Proportional Division which is obviously robust. By a numeric complete exploration of the traditional Talmudic problem of the Three Wives, we have also shown that it could be impossible to form the coalitions that violate the order of claimants (the first one with the last one, etc.). The claimants may be in the impossibility to decide of one accord what coalition must be formed, what means that it could be impossible to find a Talmudic solution to the problem of the Three Wives.

For a larger number of claimants, fulfilling the order-preserving condition may oblige to backtrack for a very large number of steps what implies an unreasonable volume of computations.

The paper has discussed also of three possible generalizations to three or more claimants of the Contested Garment method. The result is negative.

Finally, the distribution of rights between n individuals can thus be infinitely more complex than between two individuals, a case which is however often regarded as representative of the general information of the questions of allowance of rights.

7. Annex

7.1. *Monotony violated for coalition {1,3} against 2 in the Talmudic Three-Wives problem*

We consider $\mathbf{d} = (100, 200, 300)$ and $D = 600$; the results are indicated in Table 2. As the order of claimants is not preserved with the coalition 1-3, one is obliged to use the special equality rule for affecting equally the remaining awards or losses to all remaining claimants: this rule would lead to incoherencies. For example, if $E = 200$, it comes the division $(50, 100, 50)$ which is not order-preserving. The special equality rule says that the awards are shared equally between 1, 2 and 3, leading to $(66\tfrac{2}{3}, 66\tfrac{2}{3}, 66\tfrac{2}{3})$ what is not monotonic (claimant 1 has more than if $E = 250$). How to get out of this paradox? One could say that claimant 2 must not gain more than $66\tfrac{2}{3}$, in order to let the double, $133\tfrac{1}{3}$, to the two others, 1 and 3; the Talmud Division between claimants 1 and 3 gives $(50, 66\tfrac{2}{3}, 83\tfrac{1}{3})$, what is monotonic for awards and losses.

7.2. *Incoherent special equality rule for coalition {1,3} against 2 in the Talmudic Three-Wives problem*

We consider $\mathbf{d} = (100, 200, 300)$ and $D = 600$; the results are indicated in Table 3. If $E = 240$, it comes $(50, 100, 90)$ by Talmud Division, what is not order-preserving. Hence the special equality rule applied to the sharing between 1 and 3 leads to $(70, 100, 70)$ what is not order-preserving for claimant 3 and not monotonic for claimant 1; we have to go backward to apply the special equality rule to the sharing between the three claimants leading to $(80, 80, 80)$ but this is still not monotonic for claimant 1; the Talmud Division between 1 and 3 after giving 80 to claimant 2 gives $(50, 80, 110)$ what is now not monotonic for claimant 3. Hence, it is interesting to see what happens around $E = 250$ by Talmud Division if claimant 2 receives $E/3$. If E grows up to 250 from $E = 225$, at $E = 225$ it comes by Talmudic Division $\mathbf{x} = (50, 75, 100)$ then $\lim_{E \to 250^-} \mathbf{x} = (50, 83\tfrac{1}{3}, 116\tfrac{2}{3})$, which is not order preserving for claimant 3, but for $E = 250$, it comes $\mathbf{x} = (50, 100, 100)$; for $E \geq 250$ (as for $E \leq 225$), the division becomes always monotonic again. Hence, there is a clear discontinuity at $E = 250$.

How to share between 1 and 3 for $225 < E < 250$? Unfortunately, the solution *cannot* be Talmudic because the division is completely *conditioned* in this case by monotony. The awards of claimants 1 and 3 are conditioned by the monotony axiom: claimant 1 cannot receive anything else than 50 and claimant 3 anything less than 100 in order to guarantee monotonic division. For example, for $E = 240$, $x_2 = 240 - (50 + 100) = 90$ and $\mathbf{x} = (50, 90, 100)$ but we are completely outside of the Talmud Division and the orderly step-by-step process. By symmetry, the above difficulty obviously comes if the estate is large, for $350 < E < 375$ with a discontinuity at $E = 350$.

The same problem does not appear in Table 1 and Table 2, that is, for the coalitions that follow the order of claimants. For example, if we apply the Talmud Division to $E = 100$ with the coalition {2,3}, the Talmud Division would provide $(25, 50, 25)$ what violates the order: we should apply the special equality rule by sharing equally between the remaining claimants 2 and 3 what gives $(33\frac{1}{3}, 33\frac{1}{3}, 33\frac{1}{3})$; if we say that 1 must gain $33\frac{1}{3}$ to the maximum, what lets $66\frac{2}{3}$ to 2 and 3, and if we share this by Talmud Division, it comes $(33\frac{1}{3}, 33\frac{1}{3}, 33\frac{1}{3})$ also.

7.3. Monotony rule and intervals

We consider $\mathbf{d} = (100, 240, 260)$ and $D = 600$; the results are in Table 6. For $E = 200$, the Talmud Division is not order preserving for losses ($\mathbf{x} = (50, 50, 100)$). Applying the special equality rule (claimant 3 receives $E/3 = 66\frac{2}{3}$) is not order-preserving (because $\mathbf{x} = (50, 83\frac{1}{3}, 66\frac{2}{3})$) also. One remarks that by monotony, it must come anything else than $x_1 = 50$. All solutions between $\mathbf{x} = (50, 75, 75)$ and $\mathbf{x} = (50, 65, 85)$ are compatible with order-preserving and monotony rules.

Which one must be chosen in order to remain as close as possible to the spirit of the Talmud Division? One could think that claimant 3 must receive the maximum that guarantees order preservation because claimant 3 is "above" claimants 1 and 2 who form the coalition {1,2}. Unfortunately, this rule would be bad because it lacks generality: suppose that the problem appears with a coalition {1,3} against 2: who is above the other?[9] We could also say that the solution of equality between claimants 2 and 3 is a good one, in very conformity with the special equality rule. However, one claimant is outside of the coalition

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[9] We know that the problem does not appear for three claimants with this coalition, but that does not matter. Who can say what could happen for 4 claimants: how to rank a coalition {1,4} against {2,3}?

and the other inside, while the third one, claimant 1, is determined: this solution seems ad hoc.

A reasonable solution could consist into extending the special equality rule such that the middle of the interval is chosen. For $E = 200$, it comes $\mathbf{x} = (50, 70, 80)$ with coalition $\{1,2\}$. However, it corresponds to a Nash bargaining, what implies other hypotheses. Hence, we choose to keep the intervals as is in Table 6, what has the merit to show clearly the limits of the Talmud Division when coalitions are free.

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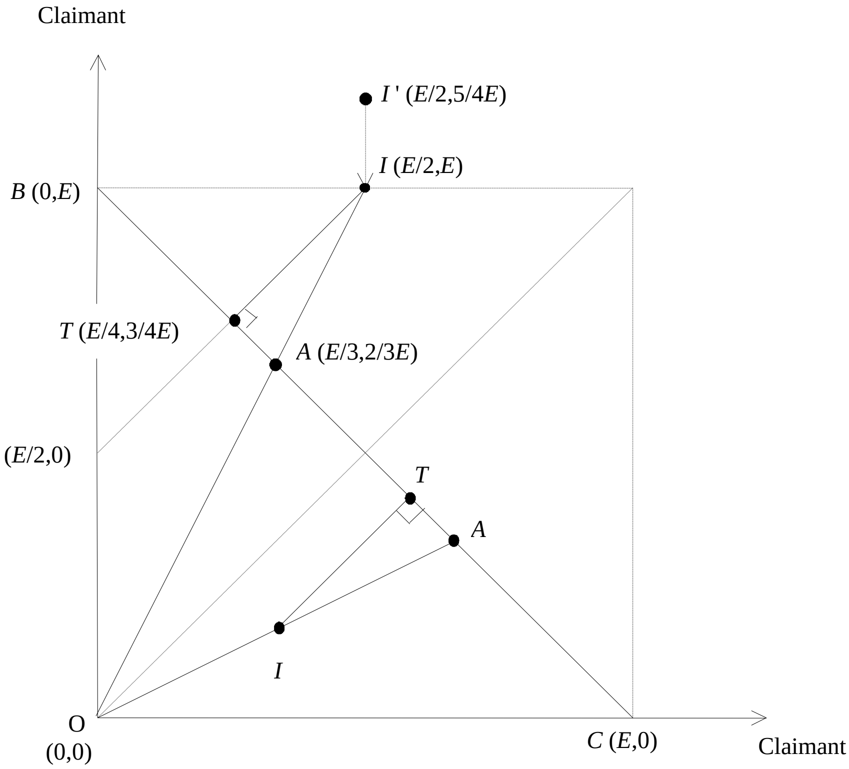

Figure 1. Example of Contested Garment

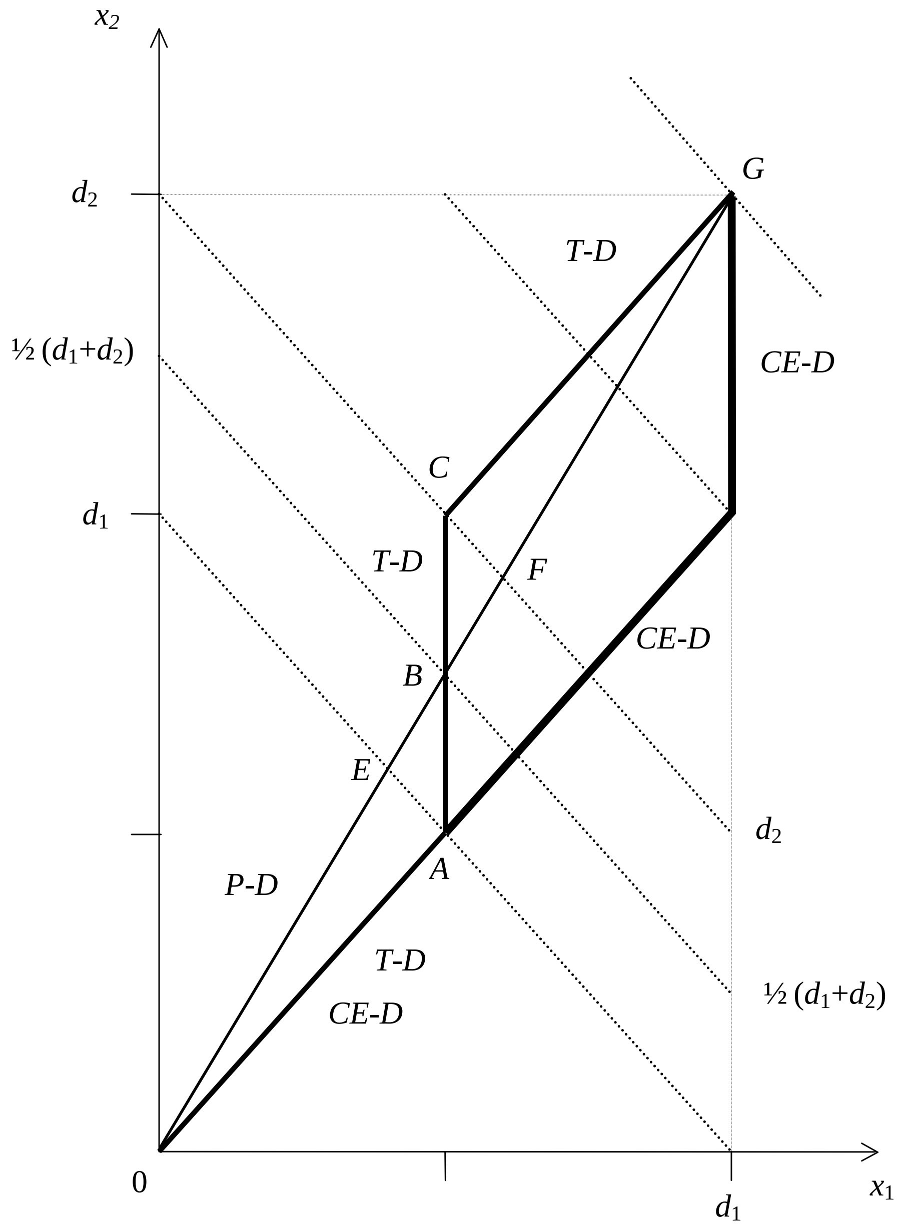

Figure 2. Division when the estate increases:
Talmud Division (T-D) and Proportional Division (P-D)
(the dotted lines are various levels of estate)

Points Distributions and total claimed	*0*	*A*	*B*	*C*	*G*
x_1	0	$\frac{1}{2}d_1$	$\frac{1}{2}d_1$	$\frac{1}{2}d_1$	d_1
x_2	0	$\frac{1}{2}d_1$	$\frac{1}{2}d_2$	$d_2-\frac{1}{2}d_1$	d_2
E	0	d_1	$\frac{1}{2}(d_1+d_2)$	d_2	d_1+d_2