On the conditions of validity of the value input-output model

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Abstract. The general aim of this paper is to show that even the simplest economic models may be largely trivial. One discusses how a value input-output model can be derived from a physical input-output model. In the physical input-output model, variables are physical quantities and prices and in the value input-output model, variables are quantities in currency units and price indexes. One demonstrates that the value model must be biperiodical, that is, never a-periodic as assumed by Leontief, and that price indexes must replace prices. Hence, two variants of the model are possible: (i) the operational model where the price indexes are the solution of the value model for the base year and (ii) the true model where the current prices solve the physical model for the current year. It is impossible to decide which model is the best. Both models diverge generally unless some strong hypotheses are made: stability of the vertically integrated physical coefficients of labor, or even stability of the physical structure itself. Working on an a-periodic value model is not innocent even if virtually all scholars and practitioners use to do by operational necessity. The results could be generalized to SAM models or to the SNA's rectangular input-output model.

Introduction

The general aim of this paper is to show that even the simplest economic models may be largely trivial by taking the example of the Leontief model.

There are two types of input-output models, the physical models and the value models. They are completely different in their purpose. In the physical input-output model, data are in physical quantities, prices are considered and fixed coefficients are defined as ratio of physical quantities (each coefficient can be greater than one); the model's solutions are outputs in physical quantities or prices. The physical model serves only as theoretical reference and can be compared to the other form of “production prices” models as the Sraffa model. It can be viewed as a simplified avatar of the Classical model of “production prices” (Ricardo, Marx, Sraffa, etc.; see Pasinetti 1977); it is not able to generate applications easily and the tables cannot be compiled from physical data.[1] Even if after defining a table in physical quantities, Leontief switches rapidly to a table in value (Leontief, 1986: 20-22),[2] while for its rectangular input-output model with two matrices, Stone (Stone, 1961; United Nations 1993, 1999; ten Raa, 2005) has never considered a physical model. Both input-output model, physical and in value, are obviously equivalent if prices are fixed (Leontief, 1985, p. 22).

On the other hand, virtually all national accounting systems or SAM (*Social Accounting Matrices*) models are based on an input-output model in value with data in currency units, price

[1] The main reason explaining why data cannot be compiled in physical quantities and are collected in currencies is very well known: it is the extreme diversity of commodities, what poses aggregation problems. It is impossible to build an interindustrial matrix from real physical data, while using data in value solves the problem of the diversity of commodities from the origin. For example, if one wants to know the physical production of cars, the car industry will answer by a list containing the number of cars produced for thousand of different models and variants: it is nonsense to add all these quantities. The same question posed in value will provide one number only, the value of this production in currency units; if one absolutely wants to know the number of cars, one will obtain the number of “mean cars” (with the mean price), that is, a mean weighted by the prices. In a word, compiled tables are always aggregated.

[2] See also Stone (1984), Miller and Blair (1985), Wheale (1985, pp. 43-48), Gale (1989).

indexes and fixed coefficients defined as ratio of currency units (each coefficient is smaller than one and the sum of all coefficients cannot exceed one). This type of models has either one square table as in the Leontief model (Leontief, 1936, 1986; Stone 1984; Miller and Blair, 1985; Wheale 1985, pp. 43-48; Gale, 1989) or two rectangular tables as in the Stone "Make-Use" model (1961) defined in the *System of National Accounts* (; United Nations 1993, 1999; ten Raa 2005). The solutions of both models are outputs in currencies or price indexes but not prices. The value models present two main advantages. 1) They are practically always used in applied studies because the data extracted from national accounting are compiled in value and thus are available only in value. 2) The value models allow homogeneity per columns and per rows. Homogeneity allows aggregating the commodities (e.g. "Steel" and "Energy") along a column of the matrix in addition to the obvious aggregation per rows. They also allow reducing the number of sectors and products by aggregation. Thus, the value model is very easy to handle.

To summarize, the physical input-output model is theoretical, the value input-output model is operational. The value square model is the core of Leontief's original contribution: it is able to open out onto real-world applications —what amply justifies the “Nobel Prize”—; it is considered from its origin by Leontief himself as going without saying. However, things are not so simple. Even if scholars and practitioners do always as if the value model was perfectly substitutable to the physical model, after deflating value tables in order to remove price effects, it is known that a value model cannot replace the corresponding physical model. (i) Unlike the value coefficients (the *techno-economic coefficients*), the physical coefficients (the *technical coefficients*) may exceed 1 and the value coefficients cannot be aggregated per columns. (ii) Even if they are widely used, by Leontief himself in its original model (1986, p. 22), the techno-economic coefficients should not be considered as the normal case. The normal case remains that of coefficients defined in physical terms with commodities and flows expressed in physical units, as when the input-output model is seen as a production function with complementary factors and is assumed in the other areas of Classic economics. (iii) Deflating tables as done in national accounting is not sufficient to replace the physical model. In the deflated tables, i.e., “tables in constant prices”, deflated quantities replace physical quantities but they remain tables in value and are not tables in physical terms. These popular arguments are annoying but they could be neglected if an additional difficulty did not come to complicate the table: reasoning in value must lead to consider a two-period model, with a *base year* and a *current year*, and to introduce price indexes. Consequently, two variants of the value model must be equally considered, an *operational model* where the price indexes solve the value model for the base year and a *theoretical model* where the current prices solve the physical model for the current year. As these variants diverge, we have to determine under what drastic conditions both models are equivalent. This is the aim of this note; to simplify the demonstrations, it discusses only the square model but it can be transposed, *mutatis mutandis*, to SAM models or to the Stone rectangular model.

Remind: the physical input-output model

It is necessary to recall how is derived the physical input-output model. We limit all the discussions to the "open" model where the final demand is exogenous and the value added the exogenous factor; however, they can be transposed to the "closed" model easily. In the physical "open" model, quantities and prices are explicitly considered. The physical quantity of commodity *i* sold by sector *i* to sector *j*, when sector *j* produces commodity *j* at year *t*, is denoted $z_{ij}^{(t)} \geq 0$ (matrix $\overline{\mathbf{Z}}^{(t)}$). The total output of commodity *i* is denoted $\bar{x}_i^{(t)}$ (vector $\overline{\mathbf{x}}^{(t)}$); $\bar{f}_i^{(t)}$ denotes the final demand of commodity *i* at year *t* (vector $\overline{\mathbf{f}}^{(t)}$); $\bar{v}_j^{(t)}$ denotes the physical value added (i.e., the quantity of labor[3]) of sector *j* at year *t* (vector $\overline{\mathbf{v}}^{(t)}$): all $\bar{v}_j^{(t)}$ must not be equal to zero at the same time. The quantities in value are respectively denoted $\mathbf{Z}^{(t)}$, $\mathbf{x}^{(t)}$, $\mathbf{f}^{(t)}$ and $\mathbf{v}^{(t)}$. The prices of commodities at year *t* are denoted $\mathbf{p}^{(t)}$ while $\mathbf{p}_v^{(t)}$ denotes the wage rate in sector *j* at year *t* (profits are considered as the owner's remuneration): one has $\mathbf{Z}^{(t)} = \hat{\mathbf{p}}^{(t)}\,\overline{\mathbf{Z}}^{(t)}$, $\mathbf{x}^{(t)} = \hat{\mathbf{p}}^{(t)}\,\overline{\mathbf{x}}^{(t)}$, $\mathbf{f}^{(t)} = \hat{\mathbf{p}}^{(t)}\,\overline{\mathbf{f}}^{(t)}$ and $\mathbf{v}^{(t)} = \hat{\mathbf{p}}_v^{(t)}\,\overline{\mathbf{v}}^{(t)}$, where the hat (e.g. in $\hat{\mathbf{p}}^{(t)}$) denotes the diagonal matrix formed from a vector (e.g. in $\mathbf{p}^{(t)}$). The model is physically closed: at year *t*, for each commodity *i*, the total of sales is equal to the total output: $\overline{\mathbf{Z}}^{(t)}\,\mathbf{s} + \overline{\mathbf{f}}^{(t)} = \overline{\mathbf{x}}^{(t)}$, where $\mathbf{s}$ denotes the sum vector $\mathbf{s}' = (1 \ldots 1)$ and the prime (e.g. in $\mathbf{s}'$) denotes the transposed of vector (e.g. in $\mathbf{s}$). The model is also closed in value; two accounting identities can be set at equilibrium:

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[3] I consider here that labor is the only factor of production, but the reader may extend it to other types of factors as Nature (and even to multiple types of factors).

(1) $\mathbf{Z}^{(t)}\,\mathbf{s}+\mathbf{f}^{(t)}=\mathbf{x}^{(t)}$

$\Leftrightarrow \overline{\mathbf{Z}}^{(t)}\,\mathbf{s}+\overline{\mathbf{f}}^{(t)}=\overline{\mathbf{x}}^{(t)}$

(2) $\mathbf{s}'\,\mathbf{Z}^{(t)}+\mathbf{v}'^{(t)}=\mathbf{x}'^{(t)}$

$\Leftrightarrow \mathbf{p}'^{(t)}\,\overline{\mathbf{Z}}^{(t)}+\mathbf{p}_v^{(t)}\,\hat{\overline{\mathbf{v}}}^{(t)}=\mathbf{p}'^{(t)}\,\hat{\overline{\mathbf{x}}}^{(t)}$

It is assumed that each sector buys each commodity in fixed proportions. Coefficients are defined in physical terms

(3) $\overline{\mathbf{A}}^{(t)}=\overline{\mathbf{Z}}^{(t)}\left[\hat{\overline{\mathbf{x}}}^{(t)}\right]^{-1}$

and it is assumed that they are stable for all *i* and *j*. Remind that nothing prevents the coefficients from being higher than 1: their magnitude depends of what scale is chosen but the determinant is scale-independent, as the result, after appropriate conversion of scale.

The economy must be at equilibrium per rows and per columns. Per rows, the accounting identity (1) becomes:

(4) $\overline{\mathbf{A}}^{(t)}\,\overline{\mathbf{x}}^{(t)}+\overline{\mathbf{f}}^{(t)}=\overline{\mathbf{x}}^{(t)}$

This solution is found in physical terms $\overline{\mathbf{x}}^{(t)}=\left[\mathbf{I}-\overline{\mathbf{A}}^{(t)}\right]^{-1}\overline{\mathbf{f}}^{(t)}$. Per columns, in the *dual*, the accounting identity (2) becomes:

(5) $\mathbf{p}'^{(t)}\,\overline{\mathbf{A}}^{(t)}+\mathbf{p}'^{(t)}_v\,\overline{\mathbf{L}}^{(t)}=\mathbf{p}'^{(t)}$

$\overline{\mathbf{L}}^{(t)}=\hat{\overline{\mathbf{v}}}^{(t)}\left[\hat{\overline{\mathbf{x}}}^{(t)}\right]^{-1}$ being the diagonal matrix of labor coefficients. So, prices come naturally as a function of the input coefficients of labor multiplied by the price of labor:

(6) $\mathbf{p}'^{(t)}=\mathbf{p}'^{(t)}_v\,\overline{\mathbf{L}}^{(t)}\left[\mathbf{I}-\overline{\mathbf{A}}^{(t)}\right]^{-1}$.

Deriving the value input-output model: difficulties

In the value "open" model physical quantities are never introduced; this model corresponds to the so-called "Leontief model". Formula (1) still holds. Coefficients are defined for any year as a ratio of values in currency units, that is:

(7) $$\mathbf{A}^{(t)} = \mathbf{Z}^{(t)} \left[\hat{\mathbf{x}}^{(t)}\right]^{-1}$$

These ratios, named *techno-economic coefficients*, are assumed stable.

Solving the primal —per rows— generates no difficulties. By introducing the techno-economic coefficients, equation (1) implies $\mathbf{A}^{(t)}\,\mathbf{x}^{(t)} + \mathbf{f}^{(t)} = \mathbf{x}^{(t)}$. The outputs in value of the year *t* are now deduced from final demands in value of the year *t*, that is, $\mathbf{x}^{(t)} = \left[\mathbf{I} - \mathbf{A}^{(t)}\right]^{-1} \mathbf{f}^{(t)}$, if the value production structure $\mathbf{A}^{(t)}$ is fixed.

In the dual, per columns, things are much more complicated. From (7) it comes:

(8) $$\mathbf{A}^{(t)} = \hat{\mathbf{p}}^{(t)}\, \overline{\mathbf{Z}}^{(t)} \left[\hat{\overline{\mathbf{x}}}^{(t)}\right]^{-1} \left[\hat{\mathbf{p}}^{(t)}\right]^{-1} = \hat{\mathbf{p}}^{(t)}\, \overline{\mathbf{A}}^{(t)} \left[\hat{\mathbf{p}}^{(t)}\right]^{-1}$$

This will be developed below.

The deadlock

If we apply the coefficients (7) directly to solve the dual, equation (2) turns into an identity, that is, $\mathbf{s}'\,\mathbf{A}^{(t)} + \mathbf{s}'\,\mathbf{L}^{(t)} = \mathbf{s}'$, which allows finding nothing. As the result is disappointing, prices could be considered. However, when using (7), the product $\mathbf{p}'^{(t)}\,\mathbf{A}^{(t)}$ is equal to $\mathbf{p}'^{(t)}\,\hat{\mathbf{p}}^{(t)}\,\overline{\mathbf{A}}^{(t)} \left[\hat{\mathbf{p}}^{(t)}\right]^{-1} = \mathbf{s}' \left[\hat{\mathbf{p}}^{(t)}\right]^{2} \overline{\mathbf{A}}^{(t)} \left[\hat{\mathbf{p}}^{(t)}\right]^{-1}$ for any year *t*, where $\left[\hat{\mathbf{p}}^{(t)}\right]^{2}$ is simply an economic nonsense: applying prices on techno-economic coefficients is not allowed.

On the other hand, price indexes can be helpful. Considering two years, what is a very usual procedure in national accounting, the *base year* (denoted 0) and the *current year* (denoted *T*),[4] one is able to define price indexes. The price indexes $\boldsymbol{\pi}$ and $\boldsymbol{\pi}_v$ are simply defined as the ratio of the prices of the current year to those of the base-year, that is $\hat{\boldsymbol{\pi}} = \hat{\mathbf{p}}^{(T)} \left[\hat{\mathbf{p}}^{(0)}\right]^{-1}$ for commodities and $\hat{\boldsymbol{\pi}}_v = \hat{\mathbf{p}}_v^{(T)} \left[\hat{\mathbf{p}}_v^{(0)}\right]^{-1}$ for labor.[5] Remark that $\boldsymbol{\pi}'\,\mathbf{Z}^{(0)} = \mathbf{p}'^{(T)}\,\overline{\mathbf{Z}}^{(0)}$: introducing the

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[4] Notice that the model remains static, with two years 0 and *T*; it is not dynamic as the founding model of Goodwin is (1952): in this model, the inputs used in year *t* are bought at the prices of year $t-1$.

[5] Laspeyres or Paasche are more sophisticated, taking into account not only the evolution of true

price indexes $\boldsymbol{\pi}$ on a model with value data of the base year —$\mathbf{Z}^{(0)}$— is mathematically the same thing than introducing the current prices $\mathbf{p}^{(T)}$ on a model with physical quantities of the base year —$\overline{\mathbf{Z}}^{(0)}$—. With price indexes, the product $\boldsymbol{\pi}'\,\mathbf{A}^{(0)}$ takes sense: $\boldsymbol{\pi}'\,\mathbf{A}^{(0)} = \mathbf{p}^{(T)}\left[\hat{\mathbf{p}}^{(0)}\right]^{-1}\hat{\mathbf{p}}^{(0)}\,\overline{\mathbf{A}}^{(0)}\left[\hat{\mathbf{p}}^{(0)}\right]^{-1} = \mathbf{p}^{(T)}\,\overline{\mathbf{A}}^{(0)}\left[\hat{\mathbf{p}}^{(0)}\right]^{-1}$. One remarks that the techno-economic coefficients are defined for the base year.

Operational model

To the question of how deriving the value model with price indexes, a first answer is: by calculating the derivative by respect to prices. It is possible to write equation (5) for the base year, that is, $\mathbf{p}'^{(0)}\,\overline{\mathbf{A}}^{(0)} + \mathbf{p}'^{(0)}_v\,\overline{\mathbf{L}}^{(0)} = \mathbf{p}'^{(0)}$. If $\overline{\mathbf{A}}^{(0)}$ and $\overline{\mathbf{L}}^{(0)}$ are stable the derivative of this last expression can be calculated: $d\mathbf{p}'^{(0)}\,\overline{\mathbf{A}}^{(0)} + d\mathbf{p}'^{(0)}_v\,\overline{\mathbf{L}}^{(0)} = d\mathbf{p}'^{(0)}$. Adding both yields:

(9) $\tilde{\mathbf{p}}'^{(T)}\,\overline{\mathbf{A}}^{(0)} + \tilde{\mathbf{p}}'^{(T)}_v\,\overline{\mathbf{L}}^{(0)} = \tilde{\mathbf{p}}'^{(T)}$

by denoting $\tilde{\mathbf{p}}^{(T)} = \mathbf{p}^{(0)} + d\mathbf{p}^{(0)}$ and $\tilde{\mathbf{p}}^{(T)}_v = \mathbf{p}^{(0)}_v + d\mathbf{p}^{(0)}_v$ the *derived current prices*. In (9), the derived current prices $\tilde{\mathbf{p}}^{(T)}$ are found from the derived current prices of labor $\tilde{\mathbf{p}}^{(T)}_v$ but also from the production structure $\overline{\mathbf{A}}^{(0)}$ and the physical structure of labor costs $\overline{\mathbf{L}}^{(0)}$ of the base year:

(10) $\tilde{\mathbf{p}}'^{(T)} = \tilde{\mathbf{p}}'^{(T)}_v\,\overline{\mathbf{L}}^{(0)}\left[\mathbf{I} - \overline{\mathbf{A}}^{(0)}\right]^{-1}$

One remarks that the prices $\tilde{p}^{(T)}_i$ are simply homothetic to the prices $p^{(0)}_i$ as only $\mathbf{p}^{(0)}_v$ has changed into $\tilde{\mathbf{p}}^{(T)}_v$. Equation (9) can be rewritten as

(11) $\tilde{\mathbf{p}}'^{(T)}\left[\hat{\mathbf{p}}^{(0)}\right]^{-1}\left[\hat{\mathbf{p}}^{(0)}\,\overline{\mathbf{A}}^{(0)}\left[\hat{\mathbf{p}}^{(0)}\right]^{-1}\right]\hat{\mathbf{p}}^{(0)} + \tilde{\mathbf{p}}'^{(T)}_v\left[\hat{\mathbf{p}}_v^{(0)}\right]^{-1}\left[\hat{\mathbf{p}}_v^{(0)}\,\overline{\mathbf{L}}^{(0)}\left[\hat{\mathbf{p}}^{(0)}\right]^{-1}\right] = \tilde{\mathbf{p}}'^{(T)}\left[\hat{\mathbf{p}}^{(0)}\right]^{-1}$

what provides the *derived price indexes*:

(12) $\tilde{\boldsymbol{\pi}}'\,\mathbf{A}^{(0)} + \tilde{\boldsymbol{\pi}}'_v\,\mathbf{L}^{(0)} = \tilde{\boldsymbol{\pi}}'$

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prices but also of physical quantities by introducing weights; see in Fisher and Shell (1997) a complete theory of production price indexes.

where $\tilde{\boldsymbol{\pi}}' = \tilde{\mathbf{p}}'^{(T)} \left[\hat{\mathbf{p}}^{(0)}\right]^{-1}$ and $\tilde{\boldsymbol{\pi}}'_v = \tilde{\mathbf{p}}'^{(T)}_v \left[\hat{\mathbf{p}}_v^{(0)}\right]^{-1}$, taking into account that $\mathbf{A}^{(0)} = \hat{\mathbf{p}}^{(0)} \overline{\mathbf{A}}^{(0)} \left[\hat{\mathbf{p}}^{(0)}\right]^{-1}$ and $\mathbf{L}^{(0)} = \hat{\mathbf{p}}_v^{(0)} \overline{\mathbf{L}}^{(0)} \left[\hat{\mathbf{p}}^{(0)}\right]^{-1}$. It is obvious that the derived price indexes are identically equal to 1 at the base year ($T = 0$): $\mathbf{s}' \mathbf{L}^{(0)} \left[\mathbf{I} - \mathbf{A}^{(0)}\right]^{-1} \equiv \mathbf{s}'$. From (12), making the hypothesis that $\mathbf{A}^{(0)}$ is stable, one deduces that the derived price indexes $\tilde{\boldsymbol{\pi}}$ are formed from the derived price index of labor, $\tilde{\boldsymbol{\pi}}_v$, but by using the value production structure and the value labor coefficients of the base year:

(13) $$\tilde{\boldsymbol{\pi}}' = \tilde{\boldsymbol{\pi}}'_v \mathbf{L}^{(0)} \left[\mathbf{I} - \mathbf{A}^{(0)}\right]^{-1}$$

This model is operational in that sense that equation (12) may be considered at the starting point instead of (9): Leontief (1936, 1985) did it but by (i) assimilating value coefficients to physical coefficients, what is rather abusive, and (ii) assimilating price indexes to prices what amounts to pose a hypothesis on the prices of the base year, as $\mathbf{p}^{(0)} = \mathbf{s}$. If we follow Leontief by posing the same hypothesis, it becomes possible to go from the value production structure to the physical production structure while it is not possible if the physical production structure is unknown.

True model

Unlike the operational model, this model is completely theoretical because it is entirely based from a physical matrix, those of year *T*. The *current true prices* —denoted $\mathbf{p}^{(T)}$— are obtained from the physical model (5) for the current year *T*:

(14) $$\mathbf{p}'^{(T)} \overline{\mathbf{A}}^{(T)} + \mathbf{p}'^{(T)}_v \overline{\mathbf{L}}^{(T)} = \mathbf{p}'^{(T)}$$

$\mathbf{p}_v^{(T)}$ being exogenous; so

(15) $$\mathbf{p}'^{(T)} = \mathbf{p}'^{(T)}_v \overline{\mathbf{L}}^{(T)} \left[\mathbf{I} - \overline{\mathbf{A}}^{(T)}\right]^{-1}$$

One remarks that equations (9) and (14) differ by the fact that $\overline{\mathbf{A}}^{(T)}$ and $\overline{\mathbf{L}}^{(T)}$ replace $\overline{\mathbf{A}}^{(0)}$ and $\overline{\mathbf{L}}^{(0)}$ respectively. The *true price indexes*, denoted $\boldsymbol{\pi}$, are deduced from the base prices and the true current prices by the formula

(16) $\boldsymbol{\pi}' = \mathbf{p}'^{(T)} \left[\hat{\mathbf{p}}^{(0)}\right]^{-1}$

Discussion: operational model vs. true model

Both models can be summarized in Figure 1. Which model is the best? We have not the answer because each one has probably its merits and the purpose of this paper is not to test them empirically; all the more, the true model is theoretical as it is based on physical data. On one hand, the true model is the right model on a theoretical point of view, on the other hand, the operational model is the only able to generate practical applications. Hence the question is: to what condition the derived price indexes and the true price indexes are equal, or equivalently, to what conditions the derived current prices are equal to the true current prices?

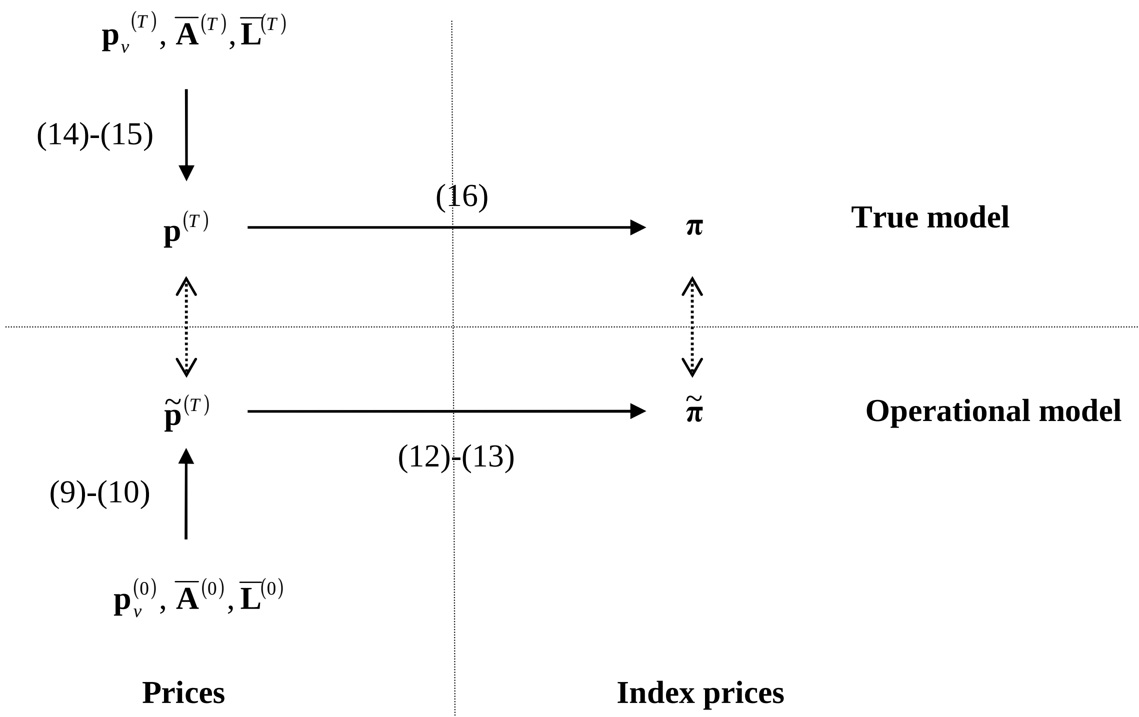

Figure 1. Comparison of derivations
(the number in parenthesis refer to the equations)

We assume that $\mathbf{p}_v^{(T)} = \tilde{\mathbf{p}}_v^{(T)}$; the price of labor being exogenous in the context of the input-out model, this is a normal hypothesis.

Theorem. Operational and true models give the same solution, that is, $\tilde{\boldsymbol{\pi}} = \boldsymbol{\pi} \Leftrightarrow \tilde{\mathbf{p}}^{(T)} = \mathbf{p}^{(T)}$, if and only if

(17) $\overline{\mathbf{L}}^{(0)} \left[\mathbf{I} - \overline{\mathbf{A}}^{(0)}\right]^{-1} = \overline{\mathbf{L}}^{(T)} \left[\mathbf{I} - \overline{\mathbf{A}}^{(T)}\right]^{-1}$

Proof. $\tilde{\boldsymbol{\pi}} = \boldsymbol{\pi} \Leftrightarrow \tilde{\mathbf{p}}^{(T)} = \mathbf{p}^{(T)} \Leftrightarrow \tilde{\boldsymbol{\pi}}'_v \, \mathbf{L}^{(0)} \left[\mathbf{I} - \mathbf{A}^{(0)}\right]^{-1} \hat{\mathbf{p}}^{(0)} = \mathbf{p}'^{(T)}_v \, \overline{\mathbf{L}}^{(T)} \left[\mathbf{I} - \overline{\mathbf{A}}^{(T)}\right]^{-1}$

$\Leftrightarrow \tilde{\boldsymbol{\pi}}'_v \left(\hat{\mathbf{p}}_v^{(0)} \, \overline{\mathbf{L}}^{(0)} \left[\hat{\mathbf{p}}^{(0)}\right]^{-1}\right) \left[\hat{\mathbf{p}}^{(0)} \left[\hat{\mathbf{p}}^{(0)}\right]^{-1} - \hat{\mathbf{p}}^{(0)} \overline{\mathbf{A}}^{(0)} \left[\hat{\mathbf{p}}^{(0)}\right]^{-1}\right]^{-1} \hat{\mathbf{p}}^{(0)} = \mathbf{p}'^{(T)}_v \, \overline{\mathbf{L}}^{(T)} \left[\mathbf{I} - \overline{\mathbf{A}}^{(T)}\right]^{-1}$

(18) $\Leftrightarrow \mathbf{p}'^{(T)}_v \, \overline{\mathbf{L}}^{(0)} \left[\mathbf{I} - \overline{\mathbf{A}}^{(0)}\right]^{-1} = \mathbf{p}'^{(T)}_v \, \overline{\mathbf{L}}^{(T)} \left[\mathbf{I} - \overline{\mathbf{A}}^{(T)}\right]^{-1}$

Equation (18) must hold for any vector $\mathbf{p}_v^{(T)}$, what is true if and only if (17) holds. •

Note that equation (17) is also equivalent to $\left[\mathbf{I} - \overline{\mathbf{A}}^{(0)}\right] \overline{\mathbf{L}}^{(0)-1} = \left[\mathbf{I} - \overline{\mathbf{A}}^{(T)}\right] \overline{\mathbf{L}}^{(T)-1}$ $\Leftrightarrow \left[\hat{\overline{\mathbf{x}}}^{(0)} - \overline{\mathbf{Z}}^{(0)}\right] \hat{\overline{\mathbf{v}}}^{(0)-1} = \left[\hat{\overline{\mathbf{x}}}^{(T)} - \overline{\mathbf{Z}}^{(T)}\right] \hat{\overline{\mathbf{v}}}^{(T)-1}$, where $\left[\hat{\overline{\mathbf{x}}}^{(t)} - \overline{\mathbf{Z}}^{(t)}\right] \mathbf{s} = \overline{\mathbf{f}}^{(t)}$ for all *t*.

The cases where only equation (17) holds can be qualified as *pseudo physical stability*. It seems complicated but we can interpret it very simply. The term *j* of the row vector $\mathbf{s}' \, \overline{\mathbf{L}}^{(t)} \left[\mathbf{I} - \overline{\mathbf{A}}^{(t)}\right]^{-1}$ is the direct and indirect quantity of labor incorporated per unit of physical output produced in sector *j* for year *t*, a concept which corresponds to the Marxian value, as underlined by Pasinetti (1977, chap. 5, annex, section 2) who calls it the vector of vertically integrated coefficients of labor.[6] This is because it holds that $\left[\mathbf{I} - \overline{\mathbf{A}}^{(t)}\right]^{-1} = \left[\mathbf{I} + \overline{\mathbf{A}}^{(t)} + \overline{\mathbf{A}}^{(t)2} + ... + \overline{\mathbf{A}}^{(t)n} + ...\right]$, that is,

(19) $\overline{\mathbf{L}}^{(t)} \left[\mathbf{I} - \overline{\mathbf{A}}^{(t)}\right]^{-1} = \overline{\mathbf{L}}^{(t)} + \overline{\mathbf{L}}^{(t)} \, \overline{\mathbf{A}}^{(t)} + \overline{\mathbf{L}}^{(t)} \, \overline{\mathbf{A}}^{(t)2} + ... + \overline{\mathbf{L}}^{(t)} \, \overline{\mathbf{A}}^{(t)n} + ...$

Hence the term $\{i, j\}$ of the matrix $\overline{\mathbf{L}}^{(t)} \left[\mathbf{I} - \overline{\mathbf{A}}^{(t)}\right]^{-1}$ is the direct and indirect quantity of labor incorporated in each input *i* per unit of physical output produced in sector *j*; in other terms, the

[6] The wording “Marxian” is not completely appropriate in the context of the Leontief model because the Marxian model is far from the Leontief model –even if both belong to the category of production prices models– but it is eloquent.

matrix $\overline{\mathbf{L}}^{(t)}\left[\mathbf{I}-\overline{\mathbf{A}}^{(t)}\right]^{-1}$ can be called the *interindustrial matrix of direct and indirect quantities of labor incorporated per unit of physical output*, that is, the *interindustrial matrix of Marxian values*. Consequently, formula (17) means that the interindustrial matrix of direct and indirect quantities of labor incorporated per unit of physical output –the interindustrial matrix of Marxian values– is stable over time.

Remark that assuming or obtaining the stability of the matrix $\overline{\mathbf{L}}^{(t)}\left[\mathbf{I}-\overline{\mathbf{A}}^{(t)}\right]^{-1}$ is much stronger than assuming or obtaining the stability of the vector $\mathbf{s'}\,\overline{\mathbf{L}}^{(t)}\left[\mathbf{I}-\overline{\mathbf{A}}^{(t)}\right]^{-1}$.

Corollary. If two of the three following properties hold, the third holds also:

- (17) holds
- $\overline{\mathbf{A}}^{(0)}=\overline{\mathbf{A}}^{(T)}$
- $\overline{\mathbf{L}}^{(T)}=\overline{\mathbf{L}}^{(0)}$

Proof. If $\overline{\mathbf{L}}^{(T)}=\overline{\mathbf{L}}^{(0)}$ and $\overline{\mathbf{A}}^{(0)}=\overline{\mathbf{A}}^{(T)}$ then (17) holds; if $\overline{\mathbf{L}}^{(T)}=\overline{\mathbf{L}}^{(0)}$ and (17) hold simultaneously, then $\left[\mathbf{I}-\overline{\mathbf{A}}^{(0)}\right]^{-1}=\left[\mathbf{I}-\overline{\mathbf{A}}^{(T)}\right]^{-1}\Leftrightarrow\overline{\mathbf{A}}^{(0)}=\overline{\mathbf{A}}^{(T)}$; if $\overline{\mathbf{A}}^{(0)}=\overline{\mathbf{A}}^{(T)}$ and (17) hold simultaneously, then $\overline{\mathbf{L}}^{(T)}=\overline{\mathbf{L}}^{(0)}$. •

It follows from the corollary that $\boldsymbol{\pi}=\tilde{\boldsymbol{\pi}}$ holds if all physical coefficients are stable: $\tilde{\boldsymbol{\pi}}=\boldsymbol{\pi}\Leftarrow\begin{cases}\overline{\mathbf{A}}^{(T)}=\overline{\mathbf{A}}^{(0)}\\ \overline{\mathbf{L}}^{(T)}=\overline{\mathbf{L}}^{(0)}\end{cases}$. Assuming that the physical technical coefficients are stable over time, i.e., $\overline{\mathbf{A}}^{(T)}=\overline{\mathbf{A}}^{(0)}$ is a very strong and rather unrealistic hypothesis. Moreover, the study of the structural change becomes completely uninteresting: the physical structural change becomes equal to zero ($\overline{\mathbf{A}}^{(T)}=\overline{\mathbf{A}}^{(0)}$ by hypothesis) and the structural change in value matrices only turns out to be a price effect.

Conclusion

Virtually all national accounting systems are based on an input-output model with data in currency units, a doubly homogenous matrix of flows (per rows as well as per columns), and price indexes: it is called the "value model". Value model does not mean a physical model with prices but a model where data are only available in currency units and where technical coefficients are derived from these data in order to have an operational model capable of using the data issued by a national accounting system. Leontief himself has considered this model as going without saying from the beginning: the aim of this note has been to demonstrate, or recall, that it is not the case. The demonstration has been based on the Leontief (square) model but it can be transposed, *mutatis mutandis*, to the Stone (rectangular) Supply-Use (or Make-Use) model advocated by the *System of National Accounts*. These critics do not concern the Sraffaian model because it remains theoretical and hence based on a physical matrix with prices.

The note has considered the physical input-output model (variables are: physical quantities and prices) and the value input-output model (variables are: quantities in currency units and price indexes). It has been demonstrated that the value input-output model is necessarily biperiodical, with a *base year* and a *current year* –what is often forgotten– and is never completely a-periodic as assumed by Leontief. As a corollary, introducing prices in the value model is a complete nonsense: considering price indexes is only allowed. This has conducted to make the distinction between two variants of the model, a *theoretical model* where the *current true prices* solve the *physical* model for the current year and an *operational model* where the *derived price indexes* are the solution of the *value* model but for the base year. It is impossible to decide which model is the best. It is a pity because they diverge generally unless some strong hypotheses are made: stability of the vertically integrated physical coefficients of labor, or even stability of the physical structure itself.

Working on an a-periodic value model is not innocent even if virtually all scholars and practitioners use to do by operational necessity. Leontief had to build an operational model: this is why he chose the simplest form –the value model– but we have shown that this choice implies making some strong assumptions on sectoral prices or on the physical coefficient matrix. Even if the input-output model was presented by Leontief as a reproduction of a simplified economy –particularly a miniature General Equilibrium– we can say ex post that the Leontief model is largely tricky: Leontief was a marvelous clever conjurer.

The adversaries of the input-output model –who think that the model is too simple, mechanical, with implausible economics– must not congratulate themselves too fast. Most of the operational models as SAMs –beyond the Stone model– use data in value and the above conclusions could probably apply to them… This conclusion could seem discouraging as the

Leontief model, of which beauty remains on its simplicity, is one of the most uncomplicated among all possible multisectoral models…

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