Poverty reduction: the paradox of the endogenous poverty line

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Abstract. When evaluating poverty, the relative poverty line may be considered as a percentage of the median income or it may be a percentage of the average income. It is proved that, with a poverty line relative to the median income, reducing poverty may become less costly in proportion to the total income as poverty increases (measured by the Sen, the Sen-Shorrocks-Thon or the Foster-Greer-Thorbecke poverty indexes) by passing from a Lorenz concentration curve to another curve associated with more poverty. This is obviously a paradox, although a largely overlooked one. However, it is shown that the paradox vanishes if the poverty line is relative to the average income. The demonstration is both experimental (algebraic or numeric where necessary) based on families of non-intersecting concentration curves produced by various algebraic functions (power function, exponential function or elliptic function but also produced by the Pareto distribution) and it is analytic. One concludes that the poverty line should be relative to the average income rather than to the median income.

Keywords. Poverty, poverty line, poverty indexes, median.

Abbreviated title. Relative poverty line paradox.

JEL Classification. D31, D63, I32.

1 INTRODUCTION

When evaluating poverty, setting a poverty line is "… also called the identification of the poor and is very important in poverty study" as Zheng, Cushing and Chow say (1995, p. 334]. The poverty line may be considered absolute: the poor are those whose income Y_i is below a given threshold of poverty $\overline{R}$: $Y_i \le \overline{R}$.[1] This is what is done in some countries, including the USA,[2] by updating the threshold of poverty mainly to adjust it for inflation.[3] Alternatively the poverty line may be considered relative.[4] Two definitions are possible. The income *R* defining the poverty level may be based on the median income *M*: the poor are those whose income is less than $\alpha\%$ of the median income; this is denoted $Y_i \le \alpha M$. While Smeeding (1991) or Blackburn (1994, 1998) take 50% of the median (or even 25%), as is done in France, the percentage $\alpha = 60\%$ is often chosen, e.g., by the European Community and OECD: the actual value may be changed but the important point is that the poverty line is linearly linked to the median income. The poverty line may be based on the average income. Jenkins and Lambert (1993) take half of the average income; Duclos and Makdissi consider a proportion of the average income (Duclos and Makdissi 2004) and indicate that the EUROMOD package uses it for comparisons across countries of the European Union. Using a relative poverty line has significant negative consequences when it comes to measuring poverty. For example, a rich country's poor may be better off than a poor country's non-poor; or the percentage of poor may remain fixed as a

[1] Actually the idea of poverty line is somewhat more complex (see Hagenaars and van Praag (1985), Ravallion (1994, 1996, 1998), and Pradhan et al. (2001)) but this is out of the scope of this paper.

[2] The U.S. Department of Health and Human Services also uses the "poverty guidelines" to determine whether a person is eligible for federal assistance.

[3] Shorrocks (1995) indicates that the poverty line in the UK "is identified with the level of benefits from income maintenance programs".

[4] Do not confuse with the idea of absolute and relative poverty measures: a relative poverty measure is scale invariant, that is, not affected by the doubling of all incomes and of the poverty line while an absolute poverty measure is translation invariant, that is, unaffected by equal increments to each income and to the poverty line (see Blackorby and Donaldson 1980), Foster and Shorrocks (1991) and Zheng (1994)).

country grows wealthier; all this is familiar enough. Even so, this is an accepted procedure because it allows international comparisons to be made.[5]

As scholars do not discuss so much what difference it makes using the median income or the average income, this paper compares the two definitions of the relative poverty. A significant question is the following. How deep is poverty relative to the country's total income and how much does it cost to eliminate poverty? It is obvious that the deeper the poverty, the more difficult it is to reduce: common sense dictates that the cost of reducing poverty should vary in the same direction as poverty. However, a contradiction may arise if poverty is deep (as evaluated by poverty measures) but the weight of poverty elimination is low relative to total income. Accordingly the cost of reducing poverty, as a percentage of total income, may be a significant variable and this raises the following question: Does it always costs more to reduce poverty if the poverty index is increasing (when passing from one concentration curve to another Lorenz-dominated curve associated with a deeper poverty) depending on the definition of the poverty line chosen. The answer to this question seems so obvious that to our knowledge no-one has ever addressed it, despite the many papers devoted to poverty measurement… To give an answer, we consider the two main families of poverty indexes: first the family of the rank-order sensitive Sen index (1976, 1979) and its derivative, particularly its more achieved form, the Sen-Shorrocks-Thon index (Shorrocks 1995); then the family of the decomposable poverty indexes of Foster-Greer-Thorbecke (1984).[6] Four typical families of concentration curves are examined so as to generate increasing poverty levels and comparing the cost of reducing poverty for the same population. These families of Lorenz curves are such that the curves are non-intersecting

[5] On the problems posed by international comparisons of poverty, see by example (Deaton 2006). For Ravallion (1992, p. 29), "Another difference between the developing country and developed countries literatures is that absolute poverty comparisons have dominated the former, while relative poverty has been more important in the latter".

[6] However, indicators based on cultural and social factors, such as the *Human Development Index* or the *Gender-Related Development Index* and the *Human Poverty Index*, all following from Sen's works on *capabilities*, are not covered by this paper. Notice than Madden (2000) has explored the interest of absolute and relative poverty lines for a study of poverty in Ireland based on the deprivation approach (there is deprivation if something lacks in household's consumption). The Watts index (1968) is left aside too (see also Muller (1998)) as well the so-called Clark index (1981) even it is a monotonic increasing function of the Chakravarty poverty index (Zheng 1997).

The paper is organized as follows: section 2 reminds what are the poverty measures considered in this paper; section 3 introduces the idea of *Relative Cost of Poverty Reduction* (RCPR); section 4 is an empirical section to demonstrate the paradoxical evolution of RCPR under the hypothesis of poverty line depending on the median income; and finally section 5 demonstrates analytically that RCPR must be paradoxically decreasing sooner or later when the Lorenz curve reaches the maximum inequality. Sections 4 and 5 demonstrate also that the paradox vanishes if the poverty line depends on the average income. Section 6 concludes.

2 REMINDER: POVERTY MEASURES

The Sen measure of poverty (Sen 1976, 1979) is written:

(1) $$S = H\left[I + (1-I)\,G_p\right] \in [0,1]$$

where:

- H is the headcount ratio, that is, the proportion of the population which is poor, with less than the poverty income R: $H = \frac{q}{n} \in [0,1]$ where $q \in N$ is the number of poor and $n \in N$ is the total number of individuals.
- I is the average poverty gap ratio of the poor. If one considers the distance to the poverty line for each poor person in the normalized Lorenz curve, I is the mean of those distances: $I = \frac{1}{q}\sum_{i=1}^{q}\frac{R-Y_i}{R} = 1 - \frac{\sum_{i=1}^{q} Y_i}{qR} \in [0,1]$, where Y_i is the income of the ith person. *It* may be considered as an inequality measure as defined by Dalton (1920).
- G_p is the Gini index of the poor, measuring the income inequality among the poor.

Formula (1) for S is valid only asymptotically, for a large number of individuals. For a small number of individuals, the formula is

(2) $$S = H\ I + \frac{q}{1+q}\,(1-I)\,G_p \in [0,1]$$

(see for example (Shorrocks 1995)); only the asymptotic form is considered here. Graphically, on a continuous normalized Lorenz concentration curve, $x \in [0,1]$ and $L(x) \in [0,1]$ are respectively the proportion of the population and of the cumulative income (see Figure 1). The poverty income generates the 'poverty line' whose slope is

(3) $$r = \frac{R}{\overline{Y}} \in R^{+}$$

where $\overline{Y}$ is the average income: $\overline{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$. H is found by identifying the point C of the normalized concentration curve $L(x)$ where the slope of the curve $L(x)$ is equal to the slope of the poverty line r. The coordinate of point C on the Y-axis is $p_1 = L(H) = \frac{\sum_{i=1}^{q} Y_i}{\sum_{i=1}^{n} Y_i} \in [0,1]$: it is the normalized aggregate income of the poor. $p_2 = \frac{qR}{\sum_{i=1}^{n} Y_i} \in [0,1]$ is the normalized aggregate income if all the poor receive the same normalized income r; graphically, it is obtained by moving along the poverty line from the origin to point F: $p_2 = H\,r$. Hence $I = \frac{p_2 - p_1}{p_2}$. The Gini index among the poor, G_p, is then equal to the area $p_1 H$ minus twice the area under the concentration curve to the left of point C, divided by $p_1 H$: $G_p = \frac{p_1\ H - 2\int_0^H L(x)\,dx}{p_1\ H} \in [0,1]$. Following Sen (1976, p. 226), S is also equal to the ratio between the curved area OCF and the triangular area OEJ ($\frac{1}{2}Hr$), as read from Figure 1.

Figure 1 about here

The Sen index has been largely criticized because it violates certain axioms or omits many desirable properties.[7] Many authors (Takayama (1979), Thon (1979, 1983), Chakravarty (1983),

[7] This paper does not go over the desirable properties or axioms to be met by the Sen poverty index (Sen 1976) and other similar or derived indexes. Following Sen (1976) and Shorrocks (1995), a good poverty index must obey certain axioms as: monotonicity (poverty increases if the income of any poor person is reduced); transfer (poverty increases if a pure

and Shorrocks (1995)) have proposed alternative indexes. In this family of indexes, the Shorrocks index is the one that meets most criteria and contains the most properties. As noted by Zheng (1997) it is the limit of the Sen index as modified by Thon (1979, 1983). Xu and Osberg (2001) call this index the Sen-Shorrocks-Thon index (denoted *SST* in what follows). This usage is followed here, even if the Shorrocks index is a particular case of the Chakravarty index proposed in 1983, as pointed out by Chakravarty himself (1997). Following Shorrocks (1995), interpreting the Sen-Shorrocks-Thon index is luminously simple; it is written $SST = \mu(z)[1+G(z)]$ where z is the distribution of normalized poverty gaps, $\mu(z)$ is the mean of this distribution and $G(z)$ is its Gini coefficient; see also (Xu and Osberg 2000). Transposed in terms of *H*, *I* and G_p, it writes as

(4) $$SST = (2-H)\,HI + H^2(1-I)\,G_p \in [0,1]$$

a formula provided by Shorrocks himself (1995), which, although less clear, does allow comparison with the Sen index. Following Xu and Osberg (2001), the Sen index is bounded above by the *SST* index while the *SST* index is a linear transformation of the Sen index: $SST = HS + 2H(1-H)I$.

The second family of indexes that is considered here is the so-called family of additively decomposable indexes which does not depend on the Gini index. The *poverty gap ratio*, alias *poverty gap index*, denoted here PGI, which is equal to $\frac{1}{n}\sum_{i=1}^{q}\frac{R-Y_i}{R} = HI$, is the value reached by *S* when all the poor have the same income (Sen's axiom N (Sen 1976)). Following (World Bank 2005a), "Poverty gap is the mean shortfall from the poverty line (counting the nonpoor as having zero shortfall), expressed as a percentage of the poverty line. This measure reflects the depth of

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transfer is made from a poor person to a richer person); replication invariance (poverty must not change if two or more identical populations are merged); subgroup consistency; homogeneity of degree zero; continuity (the value must not jump if the income of a poor person rises above the poverty income); symmetry; and normalization. Shorrocks points out that S violates the transfer and the continuity axioms. The normalization axiom plays an important role here: for Sen (1976), $S = HI$ if all the poor have the same income; constructing its index, Shorrocks (1995) relaxes this assumption by requiring that Sen's condition holds only if everyone is poor: $S = I$ whenever $H = 1$ and $G_p = 0$. However, *S* and derivated indexes still violate subgroup consistency; *S* violates replication invariance in its form (2(2) but not in its asymptotic form (1). See also (Kundu and Smith 1983) and (Pattaniak and Sengupta 1995).

poverty as well as its incidence". It is easy to demonstrate (World bank 2005b) that S is the average between the headcount ratio and the poverty gap index, weighted by the Gini index of the poor: $S = HG_p + HI(1-G_p)$. The quantity $\frac{1}{n}\sum_{i=1}^{q}\left(\frac{R-Y_i}{R}\right)^2$ is the so-called *poverty severity index*, denoted here PSI; following Foster et al. (1984), it is equal to $PSI = H\left(I^2 + (1-I)^2 C_p^2\right)$, where $C_p^2 = \sum_{i=1}^{q}\frac{(\bar{y}_p - y_i)^2}{q\,\bar{y}_p^2}$ and where $\bar{y}_p = \sum_{i=1}^{q}\frac{y_i}{q}$.[8] The quantity $\frac{1}{n}\sum_{i=1}^{q}\left(\frac{R-Y_i}{R}\right)^\beta$ is the Foster-Greer-Thorbecke index (1984), denoted here FGT; $\beta \geq 0$ is the poverty aversion parameter.[9] Subramanian (2004) reinterprets it in terms of Minkowski distance function by considering the FGT index at the power $\frac{1}{\beta}$. Notice that if $\beta = 0$ then $FGT = H$, if $\beta = 1$ then $FGT = PGI$, if $\beta = 2$ then $FGT = PSI$ and if $\beta \to \infty$, FGT is the Rawlsian measure (Aguirregabiria 2003)). On an axiomatic point of view, the FGT indexes have the advantage to be subgroup consistent, continuous, to satisfy the monotonicity axiom if $\beta > 0$ and the transfer axiom if $\beta > 1$. The family of indices of Blackorby and Donaldson (1980).

3 THE RELATIVE COST OF POVERTY REDUCTION (RCPR)

The distance from the Lorenz curve to the poverty line at the point of abscissa H is the income that must be spent to make all poor non-poor. Hence the quantity $p_2 - p_1$ is the sum that must be spent to eliminate poverty completely, as a proportion of total revenue (the curve is normalized): we call it the *Relative Cost of Poverty Reduction* (denoted RCPR). This indicator is important because it shows how much a country (or an international organization) must pay to solve the poverty problem, as a proportion of its total income. Graphically, it is equal to the distance FC in Figure 1. Notice Sen (1976, p. 223) denotes RCPR as I^* and calls it "the poverty

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[8] For Ravallion (1992 p. 39) $PSI = \frac{PGI^2}{H} + \frac{(H - PGI)^2}{H} C_p^2$ where the first term is the contribution of poverty gap and the second term is the contribution of inequality amongst the poor.

[9] On the role of the choice of β see (Tungodden 2005).

gap normalized on the total revenue of the community": $I^* = I\frac{qR}{\sum_{i=1}^{n} Y_i}$; however, he has not explored the idea further. Ravallion (1992 p. 37-38] calls this indicator "the minimum cost of eliminating poverty using targeted transfers"; he introduces also the "maximum cost of eliminating poverty without targeting" (that is, without knowing who is poor and giving *R* to everyone); he finds the ratio of the minimum cost to the maximum cost to be equal to PGI. RCPR must not be confused with the *poverty gap index*. RCPR is also the largest distance between the Lorenz curve and the poverty line (when the poverty line is above the Lorenz curve). It is not to be confused with the so-called *Robin Hood index* (or *Hoover index*), the greatest distance between the Lorenz curve and the equality line OB.

4 EXPERIMENTAL APPROACH

To study changes in RCPR relative to the poverty indexes, the concentration curve *L* may be considered a defined continuous function $y = L(x)$ instead of being deduced from a discrete real distribution of incomes. Such a procedure is not strange. For example, Kakwani and Podder (1976) (see also Raasche et al. (1980), Kakwani (1980) or Zheng (2002)) proposed a procedure for estimating a Lorenz curve based on a change in the variable, $\pi = \frac{1}{\sqrt{2}}[x + L(x)]$ and $\eta = \frac{1}{\sqrt{2}}[x - L(x)]$, giving the form:

(5) $$\eta = a\pi^{\alpha}\left(\sqrt{2} - \pi\right)^{\beta}$$

If we consider a continuum of individuals $x \in [0,1]$, the normalized concentration curve $L(x)$ must present properties which are necessary to "mimic" a normalized Lorenz curve (Raasche et al. 1980). The first property is normalization: $L(0) = 0$, $L(1) = 1$. The first derivative of the continuous Lorenz curve $l(x) = \frac{d}{dx}L(x)$ corresponds to income distribution.[10] The other

[10] Actually $l(x)$ is the distribution of gradients of income, which for the sake of convenience we take to be income, remembering that in the continuous case the incomes themselves are infinitely small by the principle of differential calculus.

properties are then $l(x) \geq 0$ (non-negative incomes) and $\frac{d^2}{dx^2}L(x) \geq 0$ (non decreasing incomes); Raasche et al. (1980) add $L(x) \leq x$. The median income is also given by the formula $m = l(\tfrac{1}{2})$. Notice that in a normalized Lorenz curve the average normalized income is fixed and always equal to 1: $\bar{l} = 1$ because $\bar{l} = \frac{1}{b-a}\int_a^b l(x)dx$ with $a = 0$ and $b = 1$.

We consider the family of fitted together and non intersecting Lorenz curves, parameterized by $k > 1$: $L_k(x)$ is such that $L_{k_1}(x) \succ L_{k_2}(x) \Leftrightarrow k_1 < k_2$ (respectively $k_1 > k_2$) where $\succ$ means "Lorenz-dominates".[11] We consider only the Lorenz curves that can evolve up to the maximum inequality. If we denote $\overline{L}(x)$ the upper limit equation of $L_k(x)$ corresponding to maximum inequality, reached when $k \to \bar{k}$ where $\bar{k}$ may be, for example, the infinite (as with the power, exponential or elliptic functions considered below) or may be 1 (for the Pareto distribution also examined later), we have $\overline{L}(x) = 0$ for $x \in [0,1[$ and $\overline{L}(x) = 1$ for $x = 1$.[12] The idea is to study all the poverty indexes, which also depend on α, as a function of *k*: $H_\alpha(k)$, $I_\alpha(k)$, $G_{p_\alpha}(k)$, $S_\alpha(k)$, $SST_\alpha(k)$, $PGI_\alpha(k)$, $PSI_\alpha(k)$ and $RCPR_\alpha(k)$. $FGT_{\alpha,\beta}(k)$ is left aside because it has two parameters but we consider that it is represented in what follows by $PGI_\alpha(k)$ and $PSI_\alpha(k)$. The question is how does $RCRP_\alpha(k)$ vary when *k* increases (respectively decreases) such that the new Lorenz curve generated by a higher (respectively lower) *k* is completely Lorenz-dominated by the preceding curve, that is, such that poverty as measured by the poverty indexes becomes greater? Does it vary in the same direction as poverty, as expected? Or does it vary in the opposite direction, which would be highly paradoxical? We will prove that

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[11] These curves must fulfill the *headcount condition* on the proportion of poor. Ravallion (1992 p. 31) underlines that if we consider two Lorenz curves *a* and *b*, *a* Lorenz dominating *b* and the mean of the distribution of *a* being larger than those of *b*, it is possible that the headcount ratio of *a* is larger than those of *b*. For Zheng (2000, p. 441]) the poverty level between two Lorenz curves *a* and *b* as measured by the Sen index is lower in *a* than in *b* if *a* Lorenz-dominates *b* under the following condition: the proportion of the poor people in *a* is not greater than in *b*; by commodity, we refer this condition as the *headcount condition*.

[12] Actually, in absolute terms, it is always possible to build a family of Lorenz curves that could never reach the maximum inequality and never tend to pass the point (1,0); for example, tending toward a straight line of equation $y = a\,x - b$, $a \geq 0$ and $b \geq 0$. However, such Lorenz curves look unnatural: they are not considered in this paper.

when the poverty line is equal to a percentage α of the median income, $RCPR_\alpha(k)$ may sometimes decrease from one income distribution to another even if $S_\alpha(k)$, $SST_\alpha(k)$, $PGI_\alpha(k)$ and $PSI_\alpha(k)$ grow, which is quite paradoxical.

Four types of Lorenz functions will be considered here: skewed to the left (power and exponential functions), neutral (elliptic function), and skewed to the right (Pareto distribution).

4.1 ENDOGENOUS POVERTY LINE (RELATIVE TO THE MEDIAN INCOME)

The slope of the poverty line is taken as a proportion α of the median income $m(k)$:

(6) $r_\alpha(k) = \alpha\, m(k)$

One can describe the poverty line as endogenous as it depends of the Lorenz curve $L_k(x)$, the median changing with this curve.

Lorenz curve generated by a power function

The first example is based on the power function:

(7) $L_k(x) = x^k$, $x \in [0,1]$, with $k > 1$[13]

This function has the requisite properties to mimic a Lorenz curve: $L_k(0) = 0$, $L_k(1) = 1$, $l_k(x) = \frac{d}{dx} L_k(x) = k\, x^{k-1} \geq 0$ and $\frac{d^2}{dx^2} L_k(x) = k\,(k-1)\, x^{k-2} \geq 0$; notice that $l_k(x)$ is itself a power function: it is equivalent to assume that the distribution of revenues is given as a power function. When k grows, (7) generates a family of successive Lorenz functions which are Lorenz-dominated and non-intersecting: $L_{k_1}(x)$ Lorenz-dominates $L_{k_2}(x)$ if and only if $k_1 < k_2$. We have $H_\alpha(k) = \frac{1}{2}\alpha^{\frac{1}{k-1}}$; and also $I_\alpha(k) = \frac{k-1}{k}$ and $G_{p_\alpha}(k) = \frac{k-1}{k+1}$: both are independent of α. The measures of poverty can be written $S_\alpha(k) = \frac{1}{2}\alpha^{\frac{1}{k-1}} \frac{(k-1)(k+2)}{k\,(k+1)}$,

[13] The power function which generates the Lorenz curve must not be confused with the so-called power law distribution $p(Z = z) = \theta\, z^{-\sigma}$, considered as close to the Pareto distribution considered further.

$SST_\alpha(k)=\alpha^{\frac{1}{k-1}}\frac{k-1}{k}\left[1-\frac{\alpha^{\frac{1}{k-1}}k}{4(k+1)}\right]$, $PGI_\alpha(k)=\frac{1}{2}\alpha^{\frac{1}{k-1}}\frac{k-1}{k}$ and $PSI_\alpha(k)=\alpha^{\frac{2k-1}{k-1}}\frac{k^{-1}+k-2}{\alpha^2(2k-1)}$. And it comes $RCPR_\alpha(k)=\alpha^{\frac{k}{k-1}}\frac{k-1}{2^k}$ with $\frac{d}{dk}RCPR_\alpha(k)=\frac{\alpha^{\frac{k}{k-1}}}{2^k}\left[1-\frac{\ln\alpha}{k-1}+(1-k)\ln 2\right]$. Figure 2 shows how evolve $S_\alpha(k)$, $SST_\alpha(k)$, $PGI_\alpha(k)$, $PSI_\alpha(k)$, $RCPR_\alpha(k)$ and also $H_\alpha(k)$, $I_\alpha(k)$ and $G_{p_\alpha}(k)$ for $\alpha=.6$ (one of the particularities of the power function is that the global Gini index, $G_\alpha(k)$, is equal to $G_{p_\alpha}(k)$); notice that the headcount condition is fulfilled as $H_\alpha(k)$ is always increasing. When the poverty measures are defined, each Lorenz curve is Lorenz-dominated by the preceding one if *k* grows and, logically, the poverty index increases as well as the proportion of poor. However, $\frac{d}{dk}RCPR(\alpha,k)<0$ for $k>k_0$ with $k_0=1.7213+.28854\sqrt{6.25-17.329\ln\alpha}$ (for example, $k_0=2.8426$ when $\alpha=.6$). Hence for $k>k_0$, $RCPR_\alpha(k)$, the percentage of total revenue that must be spent to reduce poverty, becomes decreasing with *k*, while poverty increases. Moreover, $RCPR_\alpha(k)$ tends toward 0 when $k\to\infty$! We also have $\lim_{k\to\infty}H_\alpha(k)=\frac{1}{2}$ (note that $H_\alpha(k)$ never reaches 1 and does not depend on α!), $\lim_{k\to\infty}p_{1_\alpha}(k)=0$, $\lim_{k\to\infty}p_{2_\alpha}(k)=0$, $\lim_{k\to\infty}I_\alpha(k)=1$, $\lim_{k\to\infty}G_{p_\alpha}(k)=1$, $\lim_{k\to\infty}S_\alpha(k)=\frac{1}{2}$, $\lim_{k\to\infty}SST_\alpha(k)=\frac{3}{4}$, $\lim_{k\to\infty}PGI_\alpha(k)=\frac{1}{2}$ and $\lim_{k\to\infty}PSI_\alpha(k)=\frac{1}{2}$. See Figure 2. What is worth noting is that a broader poverty measure can be associated with an easier way to reduce poverty: sometimes, the more unequal the society is (and the greater poverty is) the easier it is to reduce poverty, what is paradoxical. Graphically, in Figure 1, when *k* grows, the curved area OCF and the triangular area OEJ decrease but the ratio of the two areas (i.e. OCF:OEJ) becomes larger: *S* and poverty increase but the distance FC is shorter; see Figure 3.

Figure 2 and Figure 3 about here

On could think that this phenomenon is simply caused by the fact that the poverty line goes down when inequality increases, for a constant average revenue, and hence RCPR also decreases mechanically. However, this is only partially true. First, it is not true to say that when inequality becomes larger, the median declines always. Figure 2 shows that after $k=1$ inequality monotonically grows but the median increases for $k\in[1,1.4427]$ and $\alpha=.6$. Second, it is obviously true that the poverty line is mechanically linked to the median by equation (6) and hence follows it; but Figure 2 indicates that the maximum of the median does not coincide with the maximum of RCPR: the median of which equation is $k\,.5^{k-1}$ is growing up to $k = 1.4427$ and then it is decreasing while RCPR is increasing up to $k = 2.8426$; hence in the interval $k \in [1.4427,\, 2.8426]$, the median is decreasing but RCPR is still increasing.

Testing all the possible functions that can generate a valid Lorenz curve is beyond the scope of this paper but the one chosen here, (7), should be sufficient to serve as a counterexample, that is, as an example of a paradoxical evolution. However, certain other functions are tested here.

Lorenz curve generated by an exponential function

The concentration curve L is now an exponential function of type k^x, but a little transformed in order to fulfill the normalization conditions:

(8) $$L_k(x) = \frac{k^x - 1}{k-1} \text{ with } x \in [0,1] \text{ and } k > 1$$

Equation (8) fulfils: $L_k(0) = 0$, $L_k(1) = 1$, $l_k(x) = \frac{d}{dx} L_k(x) = k^x \frac{\ln k}{k-1} > 0$, $\frac{d^2}{dx^2} L_k(x) = k^x \frac{\ln^2 k}{k-1} \geq 0$; again, $l_k(x)$ is itself an exponential function, it is equivalent to assume that the distribution of incomes is exponential. We have found:

$$S_\alpha(k) = 1 - \frac{\frac{1}{2}\ln k}{2\ln\alpha + \ln k} + 2\frac{\ln^2\alpha - 2}{(\ln k)(2\ln\alpha + \ln k)} + \frac{2}{\alpha\sqrt{k}\ln k} + \frac{4}{\alpha\sqrt{k}(\ln k)(2\ln\alpha + \ln k)},$$

$$SST_\alpha(k) = \frac{3}{4} - \frac{1}{\ln k} - \frac{2}{\ln^2 k} + \frac{\ln\alpha}{\ln k} - \frac{\ln^2\alpha}{\ln^2 k} + \frac{2\ln\alpha}{\ln^2 k} + \frac{2}{\alpha\sqrt{k}\ln k} + \frac{2}{\alpha\sqrt{k}\ln^2 k},$$

$$PGI_\alpha(k) = \frac{\ln\alpha + \frac{1}{2}\ln k + \alpha^{-1}k^{-\frac{1}{2}} - 1}{\ln k},$$

$$PSI_\alpha(k) = \frac{2\ln\alpha + \ln k + 4\alpha^{-1}k^{-\frac{1}{2}} - \alpha^{-2}k^{-1} - 3}{2\ln k},$$

$$RCPR_\alpha(k) = \frac{\alpha\ln\alpha\sqrt{k} + \frac{1}{2}\alpha\ln k\sqrt{k} - \alpha\sqrt{k} + 1}{k-1} \text{ and}$$

$$\frac{d}{dk}RCPR_{\alpha}(k)=\frac{\frac{1}{2}\alpha\frac{\ln\alpha}{k}+\frac{1}{4}\alpha\frac{\ln k}{\sqrt{k}}}{k-1}-\frac{\alpha(\ln\alpha)\sqrt{k}+\frac{1}{2}\alpha(\ln k)\sqrt{k}-\alpha\sqrt{k}+1}{(k-1)^{2}}.$$

We again find results rather similar to those of the power function. When k grows, even if the successive Lorenz functions are Lorenz-dominated, what logically corresponds to a larger poverty measure (while the headcount condition is fulfilled because $H_{\alpha}(k)$ is always increasing), $RCPR_{\alpha}(k)$ becomes decreasing for a given value $k>k_0$ (when $\alpha=.6$, it is for $k>61.74425$, a value graphically determined because $\frac{d}{dk}RCPR_{\alpha}(k)=0$ has no analytical solutions). Figure 4 draws the poverty indicators; remark that for very small values of k (before $k=2.7777..$ for $\alpha=.6$), the evolution is paradoxical for all indicators. The paradox is again complete: if $k>61.74425$, for $\alpha=.6$, poverty elimination costs less, even if poverty tends toward a high but stabilized level! And what's more $RCPR_{\alpha}(k)$ tends toward 0 when $k\to\infty$. Again, we have also $\lim_{k\to\infty}H_{\alpha}(k)=\frac{1}{2}$ (again $H_{\alpha}(k)$ never tend to 1), $\lim_{k\to\infty}p_{1_{\alpha}}(k)=0$, $\lim_{k\to\infty}p_{2_{\alpha}}(k)=0$, $\lim_{k\to\infty}I_{\alpha}(k)=1$, $\lim_{k\to\infty}G_{p_{\alpha}}(k)=1$, $\lim_{k\to\infty}S_{\alpha}(k)=\frac{1}{2}$, $\lim_{k\to\infty}SST_{\alpha}(k)=\frac{3}{4}$, $\lim_{k\to\infty}PGI_{\alpha}(k)=\frac{1}{2}$ and $\lim_{k\to\infty}PSI_{\alpha}(k)=\frac{1}{2}$.

Figure 4 about here

Lorenz curve generated by an elliptic function in the norm $\|\ \|_k$

This function is interesting because it allows us to generate an income distribution which is symmetric about the second diagonal. It is written[14]

(9) $L_k(x)=1-\left(1-x^{k}\right)^{1/k}$, $x\in[0,1]$ and $k>1$

This Lorenz curve is derived from the elliptic function in the norm $\|\ \|_k$ $\left(\frac{y}{a}\right)^{k}+\left(\frac{x}{b}\right)^{k}=1\Leftrightarrow y=a\left(1-\left(\frac{x}{b}\right)^{k}\right)^{1/k}$ (a and b are parameters) where $a=b=1$ and it is impossible to find analytical solutions. This equation is adapted so that it fulfills the properties required to mimic a Lorenz curve: $L_k(0)=0$, $L_k(1)=1$, $l_k(x)=\frac{d}{dx}L_k(x)=\left(1-x^k\right)^{\frac{1-k}{k}}x^{k-1}>0$, $\frac{d^2}{dx^2}L_k(x)\geq 0$. However no analytical expression can be found for the tangency point between $L_k(x)$ and a parallel to the poverty line because the equation $l_k(x)=.6\,l_k(.5)$, where $l_k(x)=\frac{d}{dx}L_k(x)=\left(1-x^k\right)^{\frac{1-k}{k}}x^{k-1}$, cannot have an analytical solution in the norm $\|\ \|_k$. To overcome this, numerical simulations are conducted by considering a population of 10 000 people (a sufficiently large number for us to use formula (1), which is only valid asymptotically) and by computing either incomes or income gradients (for the same result). Figure 5 shows the results; again the headcount condition is fulfilled: $H_\alpha(k)$ is always increasing. As with the other functions, the percentage of the total income to be spent for poverty reduction, *RCPR*, decreases and tends toward zero (even if the measured poverty is always increasing).

Figure 5 about here

Pareto Distribution

The Pareto distribution is familiar enough. It is defined such that $p(Y>y)=\left(\frac{y}{y_m}\right)^{-k}$ for $k>1$, where y_m is the minimum value of income *y*. Hence its density function is written $f(y;k,y_m)=k\,\frac{y_m^k}{y^{k+1}}$ for $y>y_m$ and its cumulative distribution function as $x(y)=1-\left(\frac{y_m}{y}\right)^k \Leftrightarrow y(x)=y_m\,(1-x)^{-\frac{1}{k}}$. The Lorenz curve is classically written $L(x)=\frac{\int_{y_m}^{y(x)} y\ f(y;k,y_m)\,dy}{\int_{y_m}^{\infty} y\ f(y;k,y_m)\,dy}=\frac{\int_0^x y_m\,(1-t)^{-\frac{1}{k}}\,dt}{\int_0^1 y_m\,(1-t)^{-\frac{1}{k}}\,dt}$, which simplifies to:

[14] Mathematically, (9) is the equation of the unit ball of the norm $\|\ \|_k$; for example, $\|\ \|_2$ is the Euclidean norm and the unit ball is the circle. In $\|\ \|_k$, for $k>2$, it is impossible to find analytical solutions. The vivid wording “elliptic” is not completely satisfactory.

(10) $L_k(x) = 1 - (1-x)^{\frac{k-1}{k}}$

The family of functions (10) obviously fulfils the requisite properties: $L_k(0) = 0$, $L_k(1) = 1$, $l_k(x) = \frac{d}{dx} L_k(x) = \frac{k-1}{k}(1-x)^{-\frac{1}{k}} > 0$, $\frac{d^2}{dx^2} L_k(x) = \frac{k-1}{k^2}(1-x)^{-\frac{k+1}{k}} \geq 0$.[15] Unlike the power function and exponential function, this function is skewed to the right because, near the origin, there are "many" people in each income "level" (hence the function's other name: "the law of 80/20").[16] We have $RCPR_{r_0}(k) = \left[1 - \frac{1}{2\alpha^k}\right] \alpha \frac{k-1}{k} 2^{\frac{1}{k}} + \alpha^{1-k} \ 2^{\frac{1}{k}-1} - 1$ and

$$\frac{d}{dk} RCPR_\alpha(k) = \frac{2^{\frac{1}{k}} \alpha^{1-k}}{k^2}\left[-\frac{k(1-k)}{2}\ln 2^{-\frac{1}{k}} - \frac{k}{2}(\ln\alpha + \ln 2) + \left(\alpha^k - \frac{1}{2}\right)\left(1 - \frac{k-1}{k}\ln 2\right)\right].$$

If $\alpha < \frac{1}{2}$, the poverty measures are never defined with the Pareto distribution. For $k < -\frac{\ln 2}{\ln \alpha}$ (e.g., $k < 1.3569$ if $\alpha = .6$) the X-coordinate of the tangency point (the point where the slope of the Lorenz curve is equal to the slope of the poverty line) is negative: there are no poor ($H_\alpha(k) < 0$) and the indicators cease to be defined.[17] Notice that the successive Lorenz curves are dominated when *k* decreases, passing from infinity to 1 and not when *k* increases: $L_{k_1}(x)$ Lorenz-dominates $L_{k_2}(x)$ if and only if $k_1 > k_2$;[18] this is not a problem as the Lorenz curve is still not intersecting; the headcount condition is fulfilled: $H_\alpha(k)$ is always increasing. Hence, $RCPR_\alpha(k)$ must normally be decreasing and $\frac{d}{dk} RCPR_\alpha(k)$ negative, which is intuitive. Is this

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[15] Raasche et al. (1980) indicate that the Pareto distribution is a particular case of the Kakwani-Podder form (5) by positing $\beta = 1$ and $\alpha < 1$ (Kakwani and Podder 1980).

[16] A particular value of *k* is interesting, the one corresponding to the "80/20 law": $k = \log_4 5 = 1.161$. $H_\alpha(\log_4 5) \geq 0$ for $\alpha \geq .55044$: with the "80/20 law", poor are never found as soon as $\alpha < .55044$.

[17] Remember that the indicators are defined only for the values of *k* between 1 and the right intersect of each curve $\frac{d}{dk} RCPR_\alpha(k)$ with the X-axis.

[18] The pareto distribution considered here is inversed by respect to what is usually exposed in text books. As we consider Lorenz curves, it is the population which is in the X-axis, not the incomes, as usual for the Pareto distribution. This is why in this paper the smaller the Pareto index *k*, the smaller the proportion of rich people, unlike what is usual.

so? It all depends on the value of α. In the interval where the poverty measures are defined, $RCPR_\alpha(k)$ can be increasing as soon as $\alpha > \frac{1}{2}$. For example, if $\alpha = .6$, $RCPR_\alpha(k)$ is defined in the interval $k \in [1, 1.3569]$ but $\frac{d}{dk} RCPR_\alpha(k)$ is negative only in the interval $k \in [1.10125, 1.3569]$; hence in the interval $k \in [1, 1.10125]$, the paradox is retrieved: $RCPR_\alpha(k)$ varies in the opposite direction to poverty and $\lim_{k\to 1} RCPR_\alpha(k) = 0$; see Figure 6. We now have: $\lim_{k\to\infty} m(k) = 0$, $\lim_{k\to 1} H_\alpha(k) = 1 - \frac{1}{2\alpha} \in [0, \frac{1}{2}]$ for $\alpha \in]\frac{1}{2}, 1]$ (and again $H_\alpha(k)$ never tends to 1), $\lim_{k\to 1} p_{1_\alpha}(k) = 0$, $\lim_{k\to 1} p_{2_\alpha}(k) = 0$,

$\lim_{k\to 1} I_\alpha(k) = \frac{-1.69315 + 2\alpha - \ln\alpha}{2\alpha - 1} \in [0, .30685]$,

$\lim_{k\to 1} G_{p_\alpha}(k) = \frac{2.6931 - 2.6137\alpha + 2\alpha\ln\alpha + \ln\alpha}{2\alpha\ln\alpha + 1.3863\alpha - \ln\alpha - .69315} \in [0, .11461]$,

$\lim_{k\to 1} S_\alpha(k) = \frac{4.3863 + 2\ln\alpha + 4\alpha^2 - 8\alpha}{4\alpha^2 - 2\alpha} \in [0, .1935]$, $\lim_{k\to 1} SST_\alpha(k) = \frac{4\alpha^2 - 4\alpha + 1}{4\alpha^2} \in [0, \frac{1}{4}]$,

$\lim_{k\to 1} PGI_\alpha(k) = 1 - \frac{\ln\alpha + 1.6931}{2\alpha}$ and $\lim_{k\to 1} PSI_\alpha(k) = 1 - \frac{2.772\alpha + 4\alpha\ln\alpha + 1}{4\alpha^2}$.

Figure 6 about here

4.2 EXOGENOUS POVERTY LINE (RELATIVE TO THE AVERAGE INCOME)

It must be examined whether the paradox identified holds when the poverty revenue is relative to the average income. One can describe the poverty line as exogenous as it is now independent of the Lorenz curve: from equation (3), taking the poverty revenue as a proportion of the average income amounts to fix the slope of the poverty line; we denote it: $r = r_0$, $0 < r_0 \leq 1$.

Lorenz curve generated by a power function

If the slope of the poverty line remains exogenously fixed at an arbitrary value r_0 $0 < r_0 \leq 1$, then $S_{r_0}(k) = \frac{(k-1)(k+2)}{k+1} r_0^{\frac{1}{k-1}} k^{\frac{k}{1-k}}$, $SST_{r_0}(k) = \frac{k-1}{k(k+1)}\left(2r_0^{\frac{1}{k-1}} k^{\frac{k-2}{1-k}} + 2r_0^{\frac{1}{k-1}} k^{\frac{1}{1-k}} - r_0^{\frac{2}{k-1}} k^{\frac{k-3}{k-1}}\right)$,

$PGI_{r_0}(k) = r_0^{\frac{1}{k-1}} k^{\frac{k}{1-k}}(k-1)$, $PSI_{r_0}(k) = 2r_0^{\frac{2k-1}{k-1}} \frac{k^{\frac{k-2}{k-1}} + k^{\frac{k}{1-k}} - 2k^{\frac{1}{1-k}}}{r_0^2(2k-1)}$ and $RCPR_{r_0}(k) = r_0^{\frac{k}{k-1}}\left(k^{\frac{1}{1-k}} - k^{\frac{k}{1-k}}\right)$. For example, if $r_0 = \frac{1}{2}$ all parameters vary in a similar way to those of Figure 2, except that $RCPR_{r_0}(k)$ is now never decreasing because $\frac{d}{dk}RCPR_{r_0}(k) = 0$ for $k = r_0$ only, that is, the first derivative is never negative as $k > 1 \geq r_0$. The layout of various cases for $r_0 \in [0,1]$ indicates that $RCPR_{r_0}(k)$ is never decreasing when r_0 varies between 0 and 1 inclusive. And $\lim_{k\to\infty} RCPR_{r_0}(k) = r_0$, while $\lim_{k\to\infty} H_{r_0}(k) = 1$, $\lim_{k\to\infty} p_{1_{r_0}}(k) = 0$, $\lim_{k\to\infty} p_{2_{r_0}}(k) = r_0$, $\lim_{k\to\infty} I_{r_0}(k) = 1$, $\lim_{k\to\infty} G_{p_{r_0}}(k) = 1$, $\lim_{k\to\infty} S_{r_0}(k) = 1$, $\lim_{k\to\infty} SST_{r_0}(k) = 1$, $\lim_{k\to\infty} PGI_{r_0}(k) = 1$ and $\lim_{k\to\infty} PSI_{r_0}(k) = 1$. The paradox no longer holds…

Lorenz curve generated by an exponential function

If the poverty line is exogenously given, then

$$RCPR_{r_0}(k) = r_0 \frac{\ln r_0 + \ln(k-1) - \ln(\ln k) - 1}{\ln k} + \frac{1}{k-1} \text{ and}$$

$$\frac{d}{dk}RCPR_{r_0}(k) = r_0 \frac{\frac{1}{k-1} - \frac{1}{k\ln k}}{\ln k} - r_0 \frac{\ln r_0 + \ln(k-1) - \ln(\ln k) - 1}{k\ln^2 k} - \frac{1}{(k-1)^2}$$

The number of poor is non-negative and the poverty measures are defined only above a given value of *k*; for example, for the same $r_0 = \frac{1}{2}$, the poverty measures are defined approximately above $k = 3.51286$ (graphically determined). When the quantity $RCPR_{r_0}(k)$ is defined, it is never decreasing and $\lim_{k\to\infty} RCPR_{r_0}(k) = r_0$. Again, various numerical tests indicate that $RCPR_{r_0}(k)$ is never decreasing for $r_0 \in [0,1]$. We have $\lim_{k\to\infty} H_{r_0}(k) = 1$, $\lim_{k\to\infty} p_{1_{r_0}}(k) = 0$, $\lim_{k\to\infty} p_{2_{r_0}}(k) = r_0$, $\lim_{k\to\infty} I_{r_0}(k) = 1$, $\lim_{k\to\infty} G_{p_{r_0}}(k) = 1$, $\lim_{k\to\infty} S_{r_0}(k) = 1$, $\lim_{k\to\infty} SST_{r_0}(k) = 1$, $\lim_{k\to\infty} PGI_{r_0}(k) = 1$ and $\lim_{k\to\infty} PSI_{r_0}(k) = 1$.

Lorenz curve generated by an elliptic function in the norm $\| \ \|_k$

If *r* becomes exogenous, for example by positing $r_0 = \frac{1}{2}$, the variation returns to what one would intuitively expect: *RCPR* now rises with poverty. The simulations indicate that $RCPR_{r_0}(k)$ is never decreasing when *k* decreases for $r_0 \in [0,1]$ and $\lim_{k\to\infty} RCPR_{r_0}(k) = r_0$.

Pareto distribution

If the poverty line becomes absolute, we have $RCPR_{r_0}(k) = r_0 + \frac{r_0^{1-k}}{k-1}\left(\frac{k-1}{k}\right)^k - 1$ and $\frac{d}{dk} RCPR_{r_0}(k) = \frac{r_0^{1-k}}{k-1}\left(\frac{k-1}{k}\right)^k \left[\left(\ln\frac{k-1}{k}\right) - (\ln r_0)\right]$. Poverty measures are defined only in an interval $k \in [1, k_0]$ with $k_0 = \frac{1}{1-r_0}$ because beyond k_0, $H_{r_0}(k)$ becomes negative. When *k* decreases from k_0 to 1, $\lim_{k\to 1+} RCPR_{r_0}(k) = r_0$. For example when $r = \frac{1}{2}$, $k_0 = 2$ and $RCPR_{r_0}(k)$ always increases up to $\frac{1}{2}$ when *k* decreases. Moreover, $RCPR_{r_0}(k)$ is never decreasing when *k* decreases for $r_0 \in [0,1]$. The paradox no longer holds. We also have $\lim_{k\to\infty} H_{r_0}(k) = 1$, $\lim_{k\to\infty} p_{1_{r_0}}(k) = 0$, $\lim_{k\to\infty} p_{2_{r_0}}(k) = r_0$, $\lim_{k\to\infty} I_{r_0}(k) = 1$, $\lim_{k\to\infty} G_{p_{r_0}}(k) = 1$, $\lim_{k\to\infty} S_{r_0}(k) = 1$ and $\lim_{k\to\infty} SST_{r_0}(k) = 1$, $\lim_{k\to\infty} PGI_{r_0}(k) = 1$ and $\lim_{k\to\infty} PSI_{r_0}(k) = 1$.

5 ANALYTICAL APPROACH

5.1 POVERTY LINE RELATIVE TO THE MEDIAN INCOME

Ideally, it would be desirable to conduct a demonstration for *any* distribution or Lorenz curve. As indicated by the counter-examples when the poverty line is endogenous, RCPR may *sometimes* be decreasing; hence it cannot be demonstrated that RCPR is always decreasing (or always increasing, for that matter). However, it can be demonstrated that RCPR is either uniformly equal to zero, or it is sometimes decreasing. We denote $\overline{L}(x)$ the upper limit equation of $L_k(x)$ corresponding to the maximum inequality, reached when $k \to \bar{k}$ where $\bar{k}$ can be, for example, the infinite (as the power, exponential or elliptic functions) or can be 1 (for the Pareto distribution). We have $\bar{L}(x)=0$ for $x \in [0,1[$ and $\bar{L}(x)=1$ for $x=1$.

LEMMA. $RCPR_\alpha(k)$ tends toward zero when the Lorenz curve tends to its upper limit.

Proof. When $k \to \bar{k}$ (without reaching this limit), $\lim_{k\to\bar{k}} l_k(x)=0$ on $x \in [0,1[$ and then $\lim_{k\to\bar{k}} m(k)=0$. Hence $\lim_{k\to\bar{k}} r_\alpha(k)=0$ for any α. Even if we have no general indication of what the coordinates of the tangency point are, that is, what is the limit of $H_\alpha(k)$, we are sure that $\lim_{k\to\bar{k}} p_{1\alpha}(k)=0$ because $\bar{L}(x)=0$ for $x \in [0,1[$; and $\lim_{k\to\bar{k}} p_{2\alpha}(k)=\lim_{k\to\bar{k}}(H_\alpha(k)\, r_\alpha(k))=0$. Consequently, $\lim_{k\to\bar{k}} RCPR_\alpha(k)=0$. •

In other words, if inequality tends to infinity, the median tends to zero because all revenues (except those of the richest) and then the poverty line also, what pulls down RCPR. Notice that this lemma does not need to know the value of $H_\alpha(k)$; hence the headcount condition (Zheng 2000) does not play any role on the limit value of RCPR. Because of its limitations, the lemma does not provide any indications about the level of the poverty indexes associated with the upper limit of the Lorenz curve but it is certain that this limit corresponds to the maximum inequality as it is dominated by any other Lorenz curve. When $k \to \bar{k}$ (without reaching this limit), even if all the poor tend to have the same income, we have no general indication for $\lim_{k\to\bar{k}} I_\alpha(k)$. One must take care that even if $I_\alpha(k)=1-\frac{p_{1\alpha}(k)}{p_{2\alpha}(k)}$, where $\lim_{k\to\bar{k}} p_{1\alpha}(k)=0$ and $\lim_{k\to\bar{k}} p_{2\alpha}(k)=0$, it is false to say that $\lim_{k\to\bar{k}} I_\alpha(k)$ is always equal to zero or 1 (Cf. the experimental section). Similarly, even if the area under the curve tends to zero, we have no general indication of what $\lim_{k\to\bar{k}} G_{p_\alpha}(k)$ is: the experimental section indicates that $\lim_{k\to\bar{k}} G_{p_\alpha}(k)$ can be 1 or certain other values inside $]0,1]$. Finally, we have no general indication of what $\lim_{k\to\bar{k}} S_\alpha(k)$, $\lim_{k\to\bar{k}} SST_\alpha(k)$, $\lim_{k\to\bar{k}} PGI_\alpha(k)$, $\lim_{k\to\bar{k}} PSI_\alpha(k)$ and $\lim_{k\to\bar{k}} FGT_\alpha(k)$ are. However, all these limits can be calculated for a given Lorenz distribution.

THEOREM 1. When the poverty line is relative as a proportion of the median, either $RCPR_\alpha(k)$ is always equal to zero, or $RCPR_\alpha(k)$ is sometimes positive but it will be decreasing sooner or later and it will tend to zero anyway when inequality will tend toward its maximum.

Proof. $RCPR_\alpha(k) \geq 0$ for all k because $p_2 \geq p_1$ as $\alpha \geq 0$. If $\exists k \text{ s.t. } RCPR_\alpha(k) > 0$, $\lim_{k \to \bar{k}} RCPR_\alpha(k) = 0$ because of the Lemma: hence $RCPR_\alpha(k)$ will be decreasing sooner or later when the Lorenz curve will go toward its upper limit; if not $RCPR_\alpha(k) = 0$ for all *k*. •

This theorem is necessary because the median and RCPR are not always decreasing: following the Lemma, the median, the poverty line and RCPR tend to zero but this is only valid asymptotically. THEOREM 1 indicates what happens elsewhere: it is not guaranteed that the poverty line falls when inequality grows, as it is shown in Figure 2, Figure 4, Figure 5 and Figure 6 but the median, the poverty line and RCPR will tend to decrease. Hence, without the theorem, one is not able to say anything always true about the behavior of RCPR.

5.2 POVERTY LINE RELATIVE TO THE AVERAGE INCOME

THEOREM 2. If the slope of the poverty line remains fixed at an arbitrary value r_0, $r_0 > 0$, then $RCPR_{r_0}(k)$ tends toward r_0 while poverty is maximum.

Proof. When $\lim_{k \to \bar{k}} L_k(x) = \bar{L}(x)$ the coordinates of the tangency point are necessarily $(1,0)$ as the slope of the poverty line is such that $r_0 > 0$; hence $\lim_{k \to \bar{k}} H_{r_0}(k) = 1$. Then, $\lim_{k \to \bar{k}} p_{1r_0}(k) = 0$ but $\lim_{k \to \bar{k}} p_{2r_0}(k) = \lim_{k \to \bar{k}} (r_0 \, H_{r_0}(k)) = r_0$. Hence $\lim_{k \to \infty} RCPR_{r_0}(k) = r_0$. As $\lim_{k \to \bar{k}} I_{r_0}(k) = 1$, it comes $\lim_{k \to \bar{k}} S_{r_0}(k) = 1$, $\lim_{k \to \bar{k}} SST_{r_0}(k) = 1$, $\lim_{k \to \bar{k}} PGI_{r_0}(k) = 1$ and $\lim_{k \to \bar{k}} PSI_{r_0}(k) = 1$. •

Again, notice that the headcount condition plays no role because $H_{r_0}(k)$ always tend to 1.

6 CONCLUSION

The poverty line relative to the median income may be described as *endogenous* in contrast with the poverty line relative to the average income which may be described as *exogenous* because it is independant of the normalized distribution of incomes. This is the cause of the significant paradox which has been reported here: when the poverty line is relative to the median, a higher poverty index (the Sen or Sen-Shorrocks-Thon family of indexes but also the Foster-Greer-Thorbecke family), associated with successive dominated Lorenz curves, is compatible in many cases with a lower *Relative Cost of Poverty Reduction*. This one it is the distance from the Lorenz curve to the poverty line, that is, the income that must be spent to make all poor non-poor or the sum that must be spent to eliminate poverty completely, determined as a proportion of total revenue.

This has been demonstrated algebraically or numerically (when the equations could not be solved for mathematical reasons), (i) by directly using normalized continuous non-intersecting Lorenz curves generated from various mathematical functions (power function, elliptic function and exponential function) or (ii) by generating Lorenz curves from the Pareto distribution.[19] And it has been demonstrated analytically that the *Relative Cost of Poverty Reduction* must be decreasing sooner or later when the Lorenz curve tends toward its limit. It has also been demonstrated here that the paradox vanishes if the poverty line is relative to the average income. Why does things are right with the mean and not the median? It is because the median is endogenous. The conclusion is that the poverty line should be relative to the average income, exogenous in a normalized concentration curve, rather than to the median income, endogenous in a normalized concentration curve. This result is of course theoretical but it makes obsolete many applied studies on poverty based on the median revenue.

One could discuss to know if the RCPR is a poverty measure. Remark that nowhere in the paper it is suggested that RCPR is a poverty measure; Sen or Ravallion themselves do not suggest it. If the RCPR is considered as a poverty measure, it is deviant as its behavior is contradictory: it must be rejected. However, considering the RCPR only as a poverty measure is a misunderstanding. The RCPR is actually the amount that must be spent to alleviate poverty, that is, the counterpart of poverty. Hence, in that sense, it could be seen as a poverty measure (following the principle "everything is in everything, and reciprocally" as someone says); but a completely different one by respect to others: the RCPR corresponds to dollars to be spent (apart the fact that the RCPR is given in percentage in a normalized curve)!

Moreover, even if we assume that the RCPR is a poverty measure, the proper attitude should be the following: the mean is compatible with all poverty measures (Sen, SST, FGT and

RCPR) while the median violates one (the RCPR). This is a result in itself. One cannot assert that the RCPR has been rejected as poverty measure in the past (or not explored further as Sen did) because of its bad behavior: nobody has demonstrated this supposed bad behavior (remember: with the median, not with the mean) before this paper!

It is the median which is dubious and must be rejected; it is not the RCRR as a poverty measure that must be rejected in order to preserve the median. I have demonstrated that there is a problem with the median because of the RCPR; killing the RCPR in order to protect the median would be an inversion of reasoning and would be epistemologically strange and overcautious!

Finally, the most serious argument is the following: rejecting the RCPR as poverty measure because of its bad behavior would be a logical fault of reasoning. The RCPR has not *always* a bad behavior: it has a bad behavior only when the median is used and the right behavior —required for a good poverty measure— when the mean is used. Hence, following Descartes' principle, as all poverty indexes (S, SST, FGT) have a right behavior by respect to the median and the mean, they do not play any role. It is not the RCPR which is bad, it is the median. QED!

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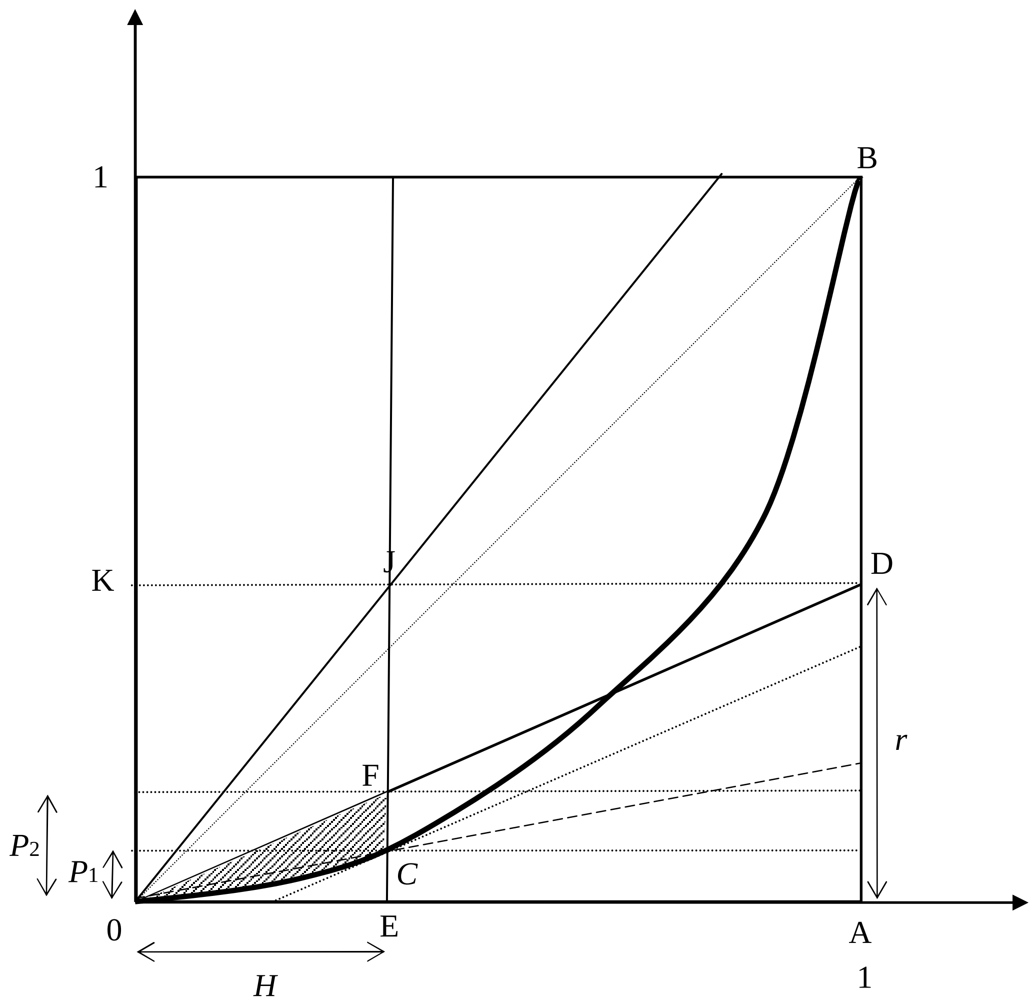

Figure 1. Graphical building of Sen's measure

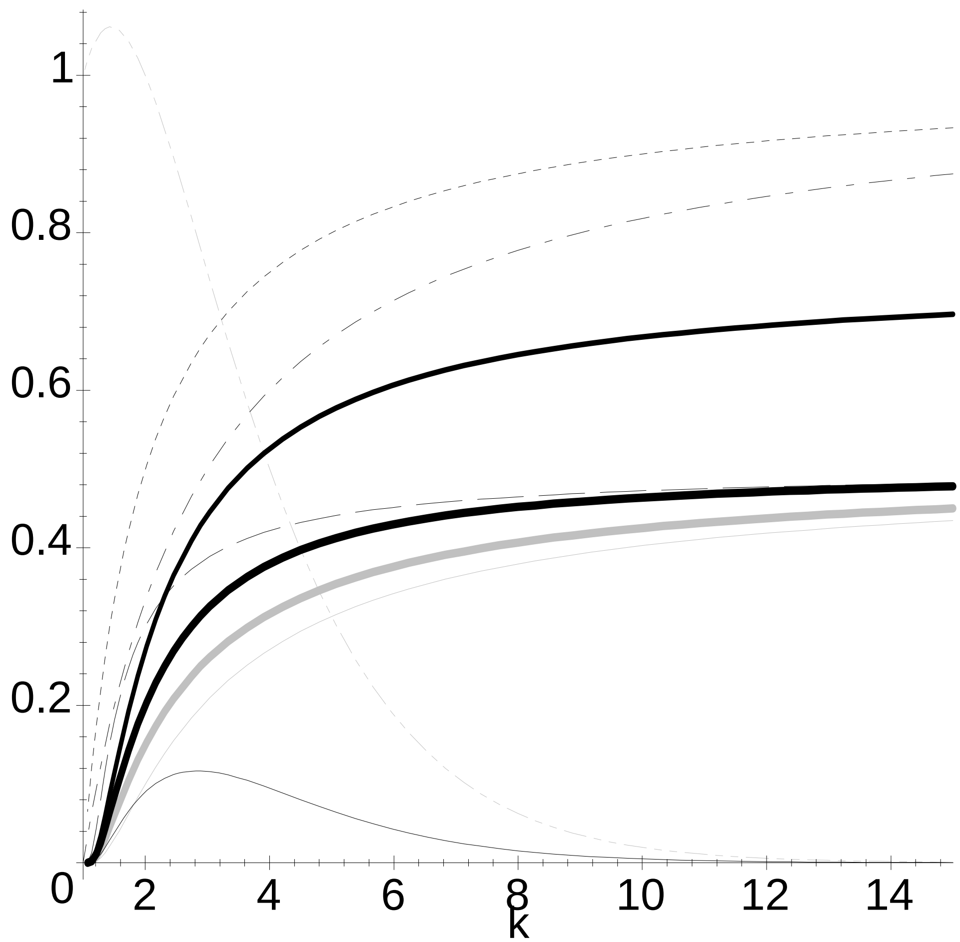

Figure 2. Poverty measures as a function of *k* for $L_k(x) = x^k$ ($\alpha = .6$)

Thick line: $S_\alpha(k)$; medium line: $SST_\alpha(k)$; thin line: $RCPR_\alpha(k)$;

dashed line: $H_\alpha(k)$; dotted line: $I_\alpha(k)$; dot-dashed line: $G_{p_\alpha}(k) = G_\alpha(k)$;

thick gray line: $PGI_\alpha(k)$; thin gray line: $PSI_\alpha(k)$;

dot-dot-dashed gray line: median

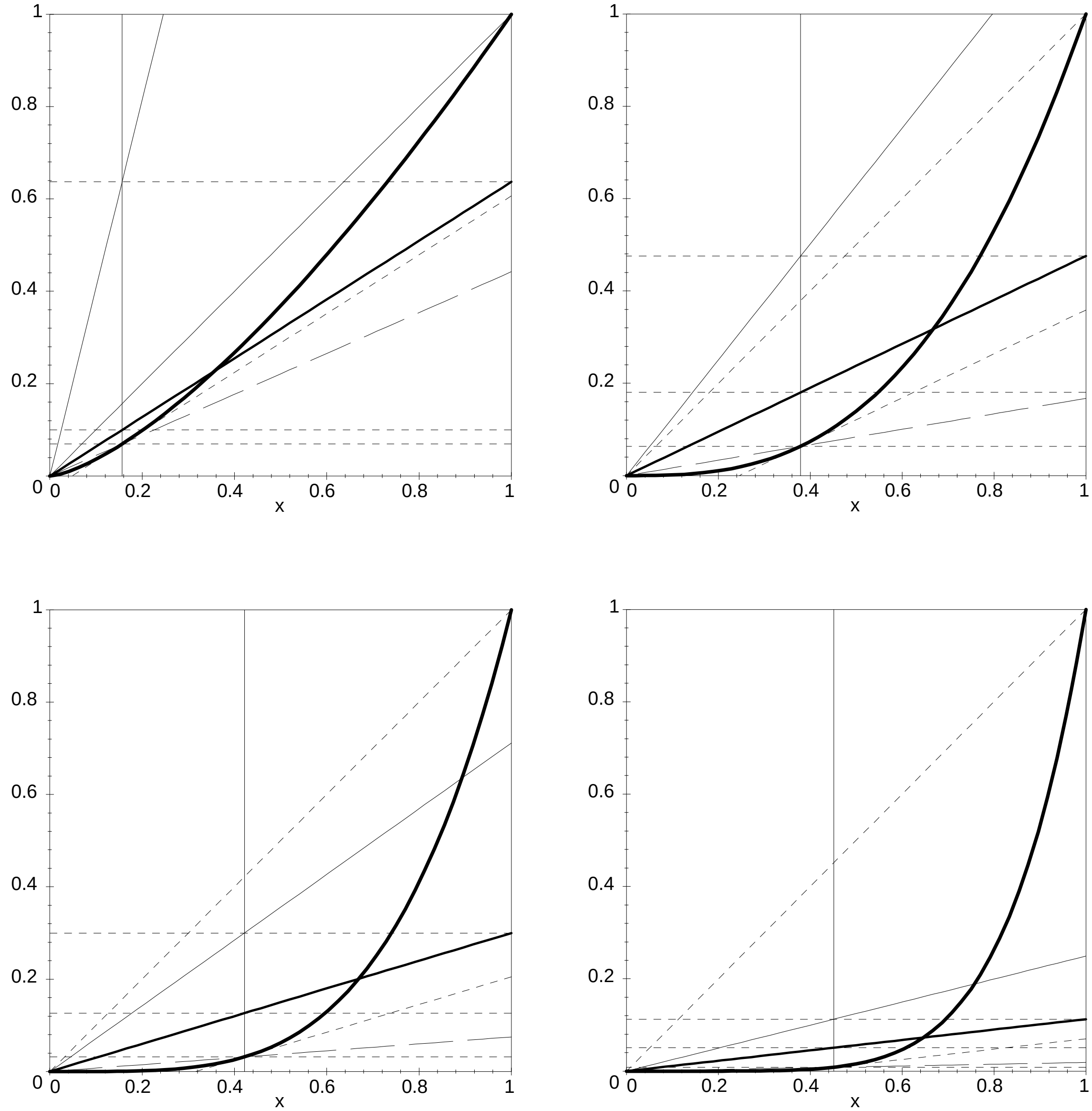

Figure 3. Graphical variation of the RCPR (power function, $\alpha = .6$)

from the upper left corner to the lower right corner:

$k = 1.4427$ **(the median is maximum at 1.0615),** $k = 2.8426$ **(RCPR is maximum),**

$k = 4$ **and** $k = 6$

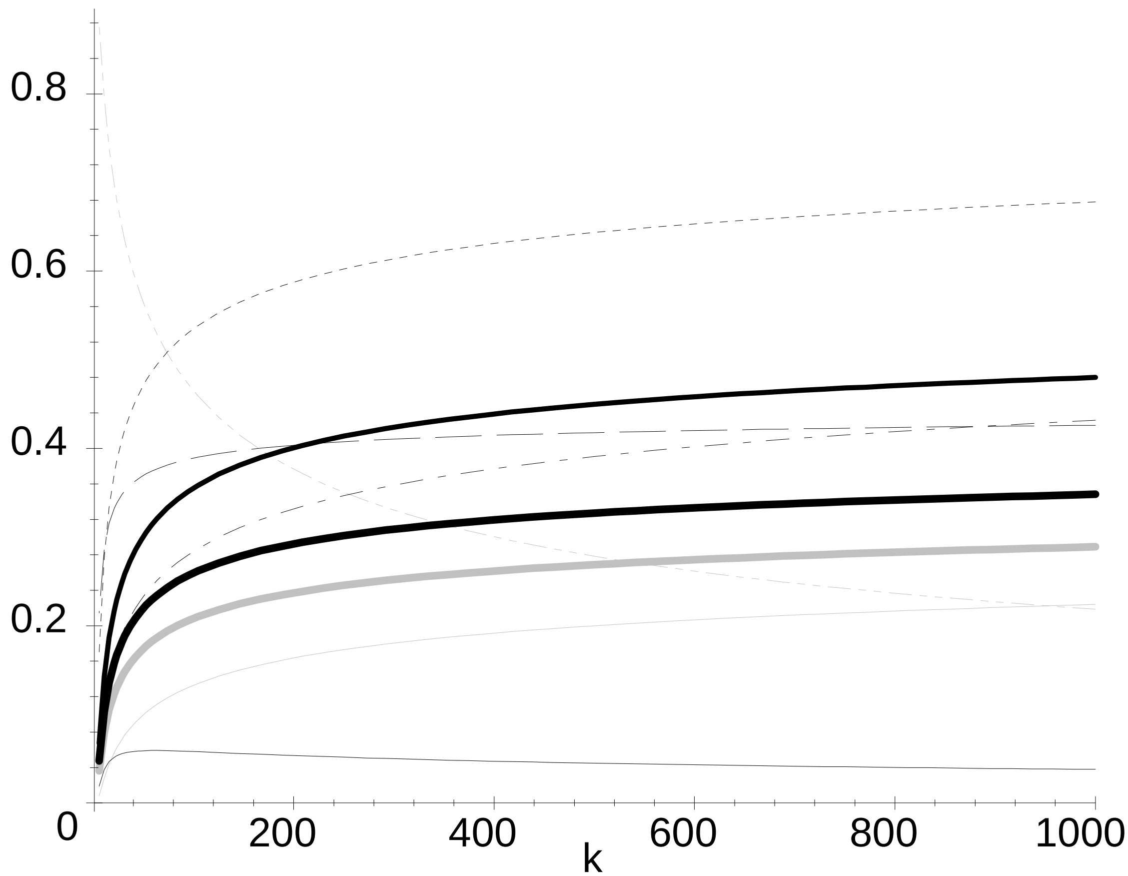

Figure 4. Poverty measures as a function of *k* for $L_k(x) = \frac{k^x - 1}{k - 1}$ **with** $k > 1$ **(**$\alpha = .6$**)**

Thick line: $S_\alpha(k)$**; medium line:** $SST_\alpha(k)$**; thin line:** $RCPR_\alpha(k)$**;**

dashed line: $H_\alpha(k)$**; dotted line:** $I_\alpha(k)$**;**

dot-dashed line: $G_{p_\alpha}(k)$**; dashed gray line:** $G_\alpha(k)$**;**

thick gray line: $PGI_\alpha(k)$**; thin gray line:** $PSI_\alpha(k)$**;**

dot-dot-dashed gray line: median

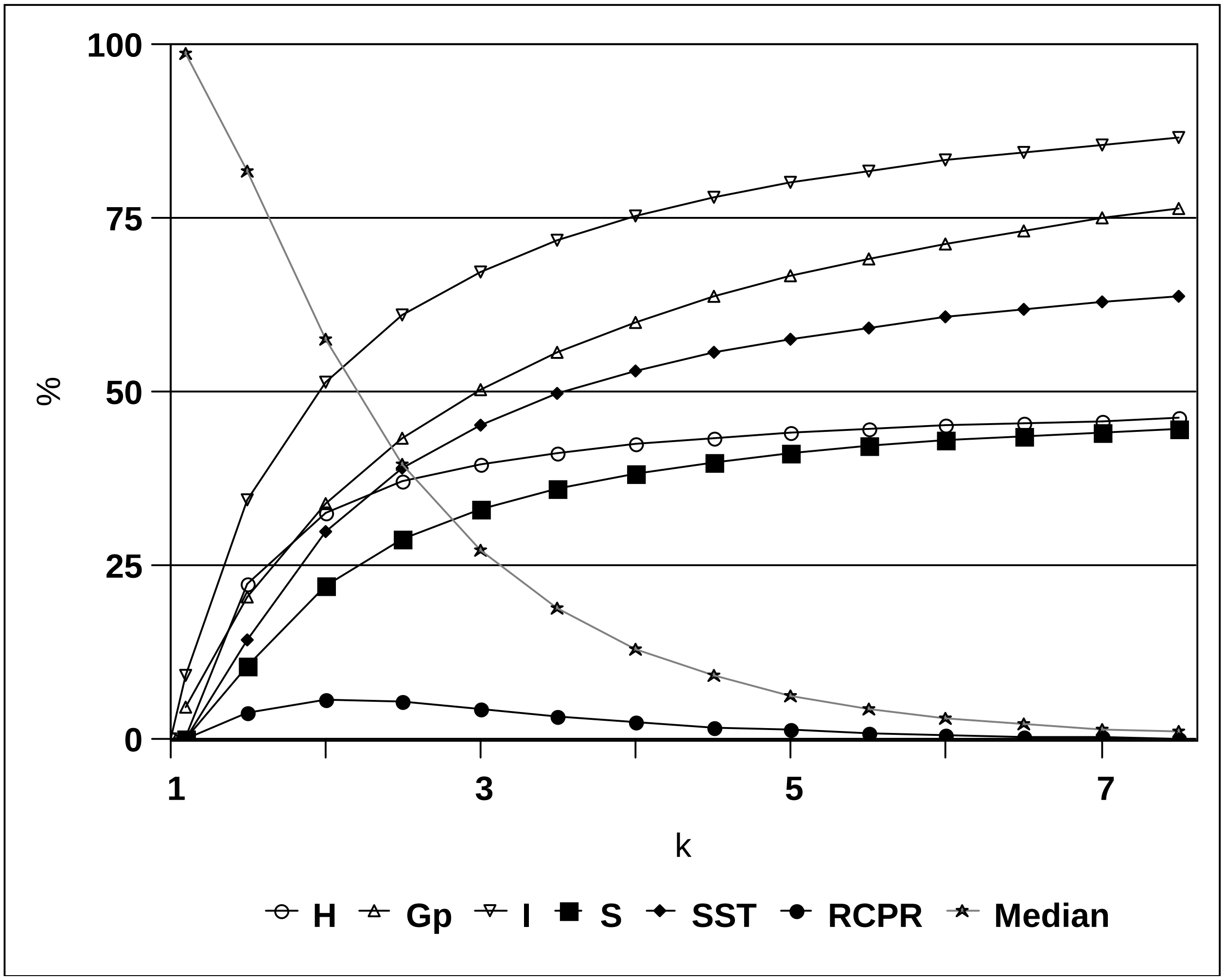

Figure 5. Poverty measures as a function of *k* for $L_k(x) = 1 - (1 - x^k)^{1/k}$ **with** $k > 1$ **(**$\alpha = .6$**)**

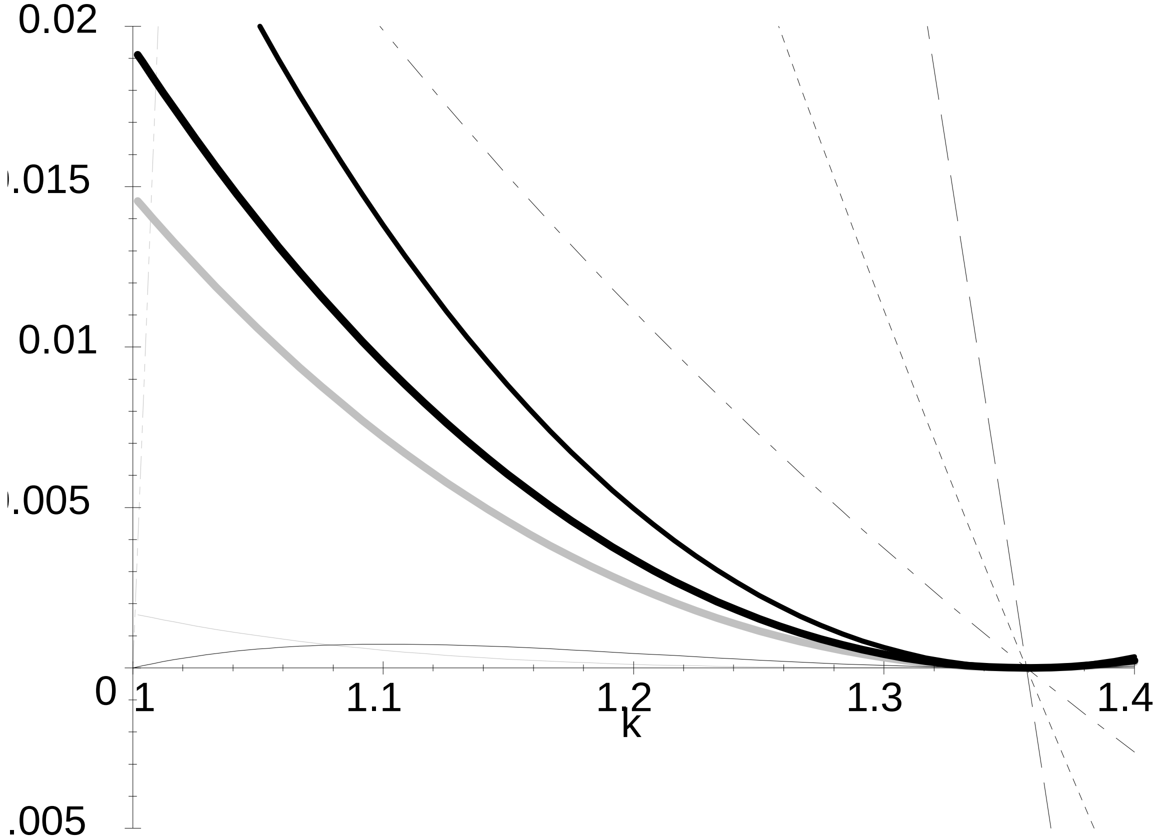

Figure 6. Poverty measures as a function of *k* ($\alpha = .6$)

for the distribution of Pareto with $k > 1$ (*k* decreasing)

Thick line: $S_\alpha(k)$; medium line: $SST_\alpha(k)$; thin line: $RCPR_\alpha(k)$;

dashed line: $H_\alpha(k)$; dotted line: $I_\alpha(k)$; dot-dashed line: $G_{p_\alpha}(k)$ ($G_\alpha(k)$ is out of range);

thick gray line: $PGI_\alpha(k)$; thin gray line: $PSI_\alpha(k)$;

dot-dot-dashed gray line: median