On the consistency of commodity-based technology in the Make-Use model

Release 2

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ABSTRACT. Most national accounting systems are based on the Make-Use model. Two hypotheses are traditionally made featuring either industry-based (IBT) or commodity-based (CBT) technologies. IBT corresponds to a consistent demand-driven model: its solution can be explained as a circuit or in probabilistic terms, even in the rectangular case. For CBT, an inverse matrix must be computed which is impossible when rectangular, fails to indicate how a commodity is distributed throughout industries and precludes interpretation of CBT as a circuit or in probabilistic terms. CBT should be interpreted as supply-driven to recover its coherence as a circuit even in the rectangular case.

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I. Introduction

Most national accounting systems around the world are based on the input-output model developed by Stone (1961) and adopted by the United Nations and OECD (United Nations, Department of Economic and Social Affairs, 1968, 1993, 1999; Lawson, 1997). Two rectangular homogenous matrices are considered:

- The Use matrix, denoted **U**, with industries as columns and commodities as rows and with final demand as a supplementary column and value added as a supplementary row; the matrix indicates how much of each commodity each industry buys in order to produce. For example, for two industries and three products:

$$\begin{matrix} \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \\ u_{31} & u_{32} \end{bmatrix} & \begin{matrix} e_1 & q_1 \\ e_2 & q_2 \\ e_3 & q_3 \end{matrix} \\ \begin{matrix} w_1 & w_2 \\ x_1 & x_2 \end{matrix} & \end{matrix}$$

- The Make matrix, denoted **V**, with industries as rows and commodities as columns, indicates how much of each commodity an industry produces. For example:

$$\begin{matrix} \begin{bmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \end{bmatrix} & \begin{matrix} x_1 \\ x_2 \end{matrix} \\ \begin{matrix} q_1 & q_2 & q_3 \end{matrix} & \end{matrix}$$

Four accounting identities are given: $x_i = \sum_{j=1}^{m} v_{ij}$ for all i, $x_j = \sum_{i=1}^{m} u_{ij} + w_j$ for all j and $q_i = \sum_{j=1}^{n} u_{ij} + e_i$ for all i, $q_i = \sum_{i=1}^{n} v_{ij}$ for all j, where w_j is the value added of industry j and e_i is the amount of commodity i sold to final demand; that is:

(1) $$\mathbf{x} = \mathbf{V}\,\mathbf{s}$$

(2) $$\mathbf{x} = \mathbf{U}'\,\mathbf{s} + \mathbf{w}$$

(3) $$\mathbf{q} = \mathbf{U}\,\mathbf{s} + \mathbf{e}$$

(4) $$\mathbf{q} = \mathbf{V}'\,\mathbf{s}$$

Technical coefficients are defined as: $a_{ij}^{u} = \dfrac{u_{ij}}{x_j}$, or,

(5) $$\mathbf{A}^{u} = \mathbf{U}\,\hat{\mathbf{x}}^{-1}$$

In this model, two alternative hypotheses about matrix **V** are posited [1]:

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[1] These two alternative models can also be combined into some mixed models (Gigantes, 1970) (ten Raa, Chakraborty and Small, 1984).

- The total output q_j of a commodity *j* is supplied by any industry *i* in fixed proportions, i.e., the *commodity-output proportion* $d_{ij} = \frac{v_{ij}}{q_j}$ is fixed (termed as *technology based on industries*), in other words the input structure of an industry does not depend on the products that it produces, that is:

  (6) $$\mathbf{D} = \mathbf{V}\,\hat{\mathbf{q}}^{-1}$$

  This hypothesis corresponds simply to a fixed market share of all industries, which may be realistic in the short run, and for Miller and Blair (1985, p. 166), it is also suitable for by-products (products whose production is linked to the main production, such as cars and automobile parts) [2].

- The total output x_i of any industry *i* is composed of commodities *j* in fixed proportions, i.e., the *industry output proportion* $c_{ij} = \frac{v_{ij}}{x_i}$ is fixed (termed as *technology based on commodities*), and the input structure of a commodity does not depend on the industry that actually produces the commodity:

  (7) $$\mathbf{C} = \hat{\mathbf{x}}^{-1}\,\mathbf{V}$$

  This hypothesis is unrealistic -- even if the System of National Accounts (SNA) of 1993 prescribes the use of commodity-based technology -- but for Miller and Blair (1985, p. 166) it is applicable to subsidiary products (secondary products -- that are primary for other sectors -- produced with the same technology as the primary product of the industry, such as automobiles and buses).

As the United Nations and OECD begin to elaborate a new set of tables for all countries after a long interruption, it is time to look again at the validity of the commodity-based model. This prompts us to take the opposite view from the supporters of the commodity-based hypothesis. This paper recalls that while the industry-based model is not scale-invariant, the commodity-based model generates negative terms. It goes on to show that: 1) because of the necessity to compute the inverse of **C**, it is impossible to interpret commodity-based technology as a circuit even in the case of square matrices, so it is economically problematic; 2) the commodity-based assumption oscillates between a demand-driven and a supply-driven model; 3) but, if it is converted into a true push-process, commodity-based technology recovers its coherence even in the rectangular case (it no longer requires computation of the inverse of **C**) and the model can be interpreted again as a circuit, which restores its coherence and removes any negative terms albeit at the cost of price and scale invariance.

[2] Even if the true by-product model is different: all secondary products are by-products and are considered as negative inputs (ten Raa, Chakraborty and Small, 1984, p. 88)

II. Reminder: discussion of the alternative models

Miller and Blair (1985, pp. 174-...) set out the complete solution of this model: each hypothesis generates two balance-accounting identities, commodities-by-commodities and industries-by-commodities and four total requirement matrices, commodities-by-commodities, commodities-by-industries, industries-by-industries and industries-by-commodities. Although they do not say so, most of the formulae require the number of commodities to equal the number of industries because the inverse of **C** and **D** have to be computed: Make and Use matrices must be square, which is a very restricting condition. However, while it is possible to generate the balance-accounting identities of industry-based technology without computing the inverse of **D**, the same is not true of commodity-based technology without computing the inverse of **C**. For industry-based technology, the commodity-by-commodity identity is found by substituting (5) in (3), that is $\mathbf{q} = \mathbf{A}^u\ \mathbf{x} + \mathbf{e}$, then by substituting (6) in (1), that is $\mathbf{x} = \mathbf{D}\ \mathbf{q}$, giving:

(8) $$\mathbf{q} = \mathbf{A}^u\ \mathbf{D}\ \mathbf{q} + \mathbf{e} \Leftrightarrow \mathbf{q} = \mathbf{A}\ \mathbf{q} + \mathbf{e}$$

by denoting $\mathbf{A} = \mathbf{A}^u\ \mathbf{D}$. By defining final demand in terms of industries' output, $\mathbf{f} = \mathbf{D}\ \mathbf{e}$, and by premultiplying (8) by **D**, the industry-by-industry identity is $\mathbf{x} = \mathbf{D}\ \mathbf{A}^u\ \mathbf{x} + \mathbf{f}$, which could be denoted $\mathbf{x} = \widetilde{\mathbf{A}}\ \mathbf{x} + \mathbf{f}$, with $\widetilde{\mathbf{A}} = \mathbf{D}\ \mathbf{A}^u$. Notice that this computation yields outputs of commodities or industries as values (price x quantity), but never quantities or prices. And there is no requirement for **U** and **V** to be square for industry-based technology.

For commodity-based technology, the commodity-by-commodity identity is found from (7) giving $\hat{\mathbf{x}} = \mathbf{V}\ \mathbf{C}^{-1}$, and by premultiplying by $\mathbf{s}'$, $\mathbf{x}' = \mathbf{s}'\ \mathbf{V}\ \mathbf{C}^{-1}$, that is, by (4), $\mathbf{x}' = \mathbf{q}'\ \mathbf{C}^{-1}$, i.e.,

(9) $$\mathbf{x} = \mathbf{C}'^{\,-1}\ \mathbf{q}$$

Once again, substituting (5) in (3) gives $\mathbf{q} = \mathbf{A}^u\ \mathbf{x} + \mathbf{e}$ and finally $\mathbf{q} = \mathbf{A}^u\ \mathbf{C}'^{\,-1}\ \mathbf{q} + \mathbf{e}$. By premultiplying this expression by $\mathbf{C}'^{\,-1}$ after redefining final demand in terms of industry output, $\mathbf{f} = \mathbf{C}'^{\,-1}\ \mathbf{e}$, the industry-by-industry identity is $\mathbf{x} = \mathbf{C}'^{\,-1} \mathbf{A}^u\ \mathbf{x} + \mathbf{f}$. Again, it will be seen that only outputs measured in value are found. Note that with commodity-based technology matrices **U** and **V** have to be square so that the inverse of **C** can be computed.

So Miller and Blair sit on the fence and do not come down in favor of either hypothesis (moreover, both hypotheses can be combined in a "mixed" hypothesis). On the other hand, for many authors (Konijn and Steenge, 1995; Kop Jansen and ten Raa, 1990) commodity-based technology is the only one "consistent with the fundamentals of input-output" (in the words of Konijn and Steenge). For example, as a column *j* of **U** (denoted $\mathbf{U}_{\bullet j}$) is the commodity input vector of industry *j*, a column *j* of $\mathbf{V}'$ (denoted $\mathbf{V}'_{\bullet j}$) is the commodity output vector of industry *j*, and **A** is the input-output coefficient matrix of dimensions commodity-by-commodity, Konijn and Steenge (1995, p. 34) say that (I apologize for the lengthy quotation):

> "For **A** to be consistent with the make and use matrices, we therefore need to have $\mathbf{U}_{\bullet j} = \mathbf{A}\ \mathbf{V}'_{\bullet j}$ for all *j* or $\mathbf{U} = \mathbf{A}\ \mathbf{V}'$. The input-output matrix **A** has to fulfill this equation to achieve full consistency between the data in the make, use and input-output matrices, and the way that the

input-output matrix will be used in input-output analysis. The attentive reader will have noticed that this fundamental equation is equal to the equation of the commodity technology model. Thus, we have shown that the commodity technology model is the only model that is consistent with the fundamentals of input-output.".

The above argument -- "we therefore need to have $\mathbf{U}_{\bullet j} = \mathbf{A}\,\mathbf{V}'_{\bullet j}$ for all j" -- is not sufficiently convincing in itself (even if it *might* possibly not be wrong) because the opposite argument, developed for industry-based technology, could be correct (even if more complicated): commodities are produced by industries in accordance with the commodity-output-coefficients d_{ij} (found from the Make matrix), and industries demand products by means of technical coefficients a^u_{ij} (found from the Use matrix). That is, the coefficients of **A** are found by the product of two matrices of coefficients, the sub-Markovian matrix of technical coefficients, $\mathbf{A}^u$, and the Markovian matrix of commodity-output-proportions **D**, according to (8) [3]:

(10) $$\mathbf{A} = \mathbf{A}^u\,\mathbf{D} = (\mathbf{U}\,\hat{\mathbf{x}}^{-1})\,(\mathbf{V}\,\hat{\mathbf{q}}^{-1}) = \mathbf{U}\,\langle\mathbf{V}\,\mathbf{s}\rangle^{-1}\,\mathbf{V}\,\langle\mathbf{V}'\,\mathbf{s}\rangle^{-1}$$

Industry-based technology does not fully comply with a set of four axioms, unlike commodity-based technology (Kop Jansen and ten Raa, 1990), namely (**A** being a function of **U** and **V**: $\mathbf{A} = \mathbf{A}(\mathbf{U},\ \mathbf{V})$):

1) the material balance: $\mathbf{q} = \mathbf{A}(\mathbf{U},\ \mathbf{V})\,\mathbf{q} + \mathbf{e}$ -- that is (8) -- which transforms into $\mathbf{A}(\mathbf{U},\ \mathbf{V})\,\mathbf{V}'\,\mathbf{s} = \mathbf{U}\,\mathbf{s}$ by using (3) and (4);

2) the financial balance: $\mathbf{p}'\,\mathbf{V}' = \mathbf{p}'\,\mathbf{A}(\mathbf{U},\ \mathbf{V})\,\mathbf{V}' + \mathbf{w}'$, which transforms into $\mathbf{A}(\mathbf{U},\ \mathbf{V})\,\mathbf{V}'\,\mathbf{s} = \mathbf{U}\,\mathbf{s}$ by using (2) and (1) after positing $\mathbf{p} = \mathbf{s}$ [4];

3) price invariance: $\mathbf{A}(\hat{\mathbf{p}}\,\mathbf{U},\ \mathbf{V}\,\hat{\mathbf{p}}) = \hat{\mathbf{p}}\,\mathbf{A}(\mathbf{U},\ \mathbf{V})\,\hat{\mathbf{p}}^{-1}$ for all $\mathbf{p} > \mathbf{0}$;

4) scale invariance: $\mathbf{A}\left(\mathbf{U}\,\hat{\mathbf{k}},\ \hat{\mathbf{k}}\,\mathbf{V}\right) = \mathbf{A}(\mathbf{U},\ \mathbf{V})$ for all $\mathbf{k} > \mathbf{0}$.

Financial balance holds for the commodity model: $\mathbf{s}'\,\mathbf{A}(\mathbf{U},\ \mathbf{V})\,\mathbf{V}' = \mathbf{s}'\,\mathbf{U}\,\mathbf{V}'^{-1}\,\mathbf{V}' = \mathbf{s}'\,\mathbf{U}$. But, contrary to the assertion of Kop Jansen and ten Raa (1990, p. 218, Table 1), it does hold for the industry model: $\mathbf{s}'\,\mathbf{U} = \mathbf{s}'\,\mathbf{A}^u\,\hat{\mathbf{x}} = \mathbf{s}'\,(\mathbf{A}^u\,\mathbf{D})\,\mathbf{V}' = \mathbf{s}'\,\mathbf{A}(\mathbf{U},\ \mathbf{V})\,\mathbf{V}'$ because $\hat{\mathbf{x}} = \mathbf{V}\,\mathbf{D}' = \mathbf{D}\,\mathbf{V}'$; it is easy to demonstrate this starting from (1), remembering that $\mathbf{D}'\,\mathbf{s} = \mathbf{s}$: $\mathbf{x} = \mathbf{V}\,\mathbf{s} \Leftrightarrow \mathbf{x} = \mathbf{V}\,\mathbf{D}'\,\mathbf{s} \Leftrightarrow \hat{\mathbf{x}}\,\mathbf{s} = \mathbf{V}\,\mathbf{D}'\,\mathbf{s} \Leftrightarrow \mathbf{s} = \hat{\mathbf{x}}^{-1}\,\mathbf{V}\,\mathbf{D}'\,\mathbf{s} \Leftrightarrow \mathbf{V}\,\mathbf{D}'\,\mathbf{s} = (\mathbf{V}\,\mathbf{D}'\,\hat{\mathbf{x}}^{-1})\,\mathbf{V}\,\mathbf{D}'\,\mathbf{s}$, thus: $\mathbf{V}\,\mathbf{D}'\,\hat{\mathbf{x}}^{-1} = \mathbf{I}$. In fact, the "financial balance" indicates only that (2) holds, that is an equilibrium by columns, while the "material balance" corresponds to an equilibrium by rows.

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[3] This formula, was found in (Miller and Blair, 1985, p.173) but it works even for rectangular matrices.

[4] Axioms 2 and 4 can be criticized because, as said above, identities (1), (2) (3) and (4) require homogeneity by row and columns - all matrices can be summed by rows and columns -- , so all terms are expressed as values (product of a quantity by a price), as well as coefficients: *stricto sensu*, there are no prices in the model . So, to pre- and postmultiply **A** by prices in the price-invariance formula is heroic (even done by Stone...).

Price invariance applies for the most simple case of commodity-based technology:

$$\mathbf{A}(\hat{\mathbf{p}}\,\mathbf{U},\ \mathbf{V}\,\hat{\mathbf{p}}) = \hat{\mathbf{p}}\,\mathbf{U}\,(\mathbf{V}\,\hat{\mathbf{p}})'^{\,-1} = \hat{\mathbf{p}}\,\mathbf{U}\,\mathbf{V}'^{\,-1}\,\hat{\mathbf{p}}^{-1} = \hat{\mathbf{p}}\,\mathbf{A}(\mathbf{U},\ \mathbf{V})\,\hat{\mathbf{p}}^{-1},$$

but it does not hold for the case of industry-based technology:

$$\mathbf{A}\left(\hat{\mathbf{p}}\,\mathbf{U},\ \mathbf{V}\,\hat{\mathbf{p}}\right) = \hat{\mathbf{p}}\,\mathbf{U}\,\langle\mathbf{V}\,\hat{\mathbf{p}}\,\mathbf{s}\rangle^{-1}\,\mathbf{V}\,\hat{\mathbf{p}}\,\left\langle(\mathbf{V}\,\hat{\mathbf{p}})'\,\mathbf{s}\right\rangle^{-1} \neq \hat{\mathbf{p}}\,\mathbf{U}\,\langle\mathbf{V}\,\mathbf{s}\rangle^{-1}\,\mathbf{V}\,\langle\mathbf{V}'\,\mathbf{s}\rangle^{-1}\,\hat{\mathbf{p}}^{-1} = \hat{\mathbf{p}}\,\mathbf{A}(\mathbf{U},\ \mathbf{V})\,\hat{\mathbf{p}}^{-1}$$

even if **D** does not depend on prices:

$$\mathbf{D}\left(\hat{\mathbf{p}}\,\mathbf{U},\ \mathbf{V}\,\hat{\mathbf{p}}\right) = \mathbf{V}\,\hat{\mathbf{p}}\,\langle\hat{\mathbf{p}}\,\mathbf{V}'\,\mathbf{s}\rangle^{-1} = \mathbf{V}\,\hat{\mathbf{p}}\,\hat{\mathbf{p}}^{-1}\,\langle\mathbf{V}'\,\mathbf{s}\rangle^{-1} = \mathbf{V}\,\langle\mathbf{V}'\,\mathbf{s}\rangle^{-1} = \mathbf{D}(\mathbf{U},\ \mathbf{V})$$

because, if **q** and **r** are two vectors, $\langle\hat{\mathbf{q}}\,\mathbf{r}\rangle = \hat{\mathbf{q}}\,\hat{\mathbf{r}} = \hat{\mathbf{r}}\,\hat{\mathbf{q}}$.

The industry-based model is not scale invariant:

$$\mathbf{A}\left(\mathbf{U}\,\hat{\mathbf{k}},\ \hat{\mathbf{k}}\,\mathbf{V}\right) = \mathbf{U}\,\hat{\mathbf{k}}\left\langle\hat{\mathbf{k}}\,\mathbf{V}\,\mathbf{s}\right\rangle^{-1}\hat{\mathbf{k}}\,\mathbf{V}\left\langle\left(\hat{\mathbf{k}}\,\mathbf{V}\right)'\mathbf{s}\right\rangle^{-1} \neq \mathbf{U}\,\langle\mathbf{V}\,\mathbf{s}\rangle^{-1}\,\mathbf{V}\,\langle\mathbf{V}'\,\mathbf{s}\rangle^{-1} = \mathbf{A}(\mathbf{U},\ \mathbf{V})$$

even if $\mathbf{A}^u$ is scale invariant: $\mathbf{A}^u\left(\mathbf{U}\,\hat{\mathbf{k}},\ \hat{\mathbf{k}}\,\mathbf{V}\right) = \mathbf{U}\,\hat{\mathbf{k}}\left\langle\hat{\mathbf{k}}\,\mathbf{V}\,\mathbf{s}\right\rangle^{-1} = \mathbf{U}\,\hat{\mathbf{k}}\ \hat{\mathbf{k}}^{-1}\,\langle\mathbf{V}\,\mathbf{s}\rangle^{-1} = \mathbf{U}\,\langle\mathbf{V}\,\mathbf{s}\rangle^{-1}$

while the commodity-based model is:

$$\mathbf{A}\left(\mathbf{U}\,\hat{\mathbf{k}},\ \hat{\mathbf{k}}\,\mathbf{V}\right) = \mathbf{U}\,\hat{\mathbf{k}}\left(\hat{\mathbf{k}}\,\mathbf{V}\right)'^{-1} = \mathbf{U}\,\hat{\mathbf{k}}\,\hat{\mathbf{k}}^{-1}\,\mathbf{V}'^{\,-1} = \mathbf{U}\,\mathbf{V}'^{\,-1} = \mathbf{A}(\mathbf{U},\ \mathbf{V}).$$

III. The broken circuit under the commodity-based hypothesis

The Make-Use model can be interpreted in terms of circuit (see a similar demonstration for the Leontief model in the appendix). Everything is dependent on the plausibility of the circular process as described by the alternative hypotheses: either the process is plausible and the solution of the model is economically meaningful, or it is not. Consider a surge of final demand $\Delta e_j^{(0)}$ for commodity j is an equal need for commodity j: $\Delta q_j^{(0)} = \Delta e_j^{(0)}$. This generates an increase in the production of industry i: $\Delta x_i^{(1)} = d_{ij}\,\Delta q_j^{(0)}$; so, in total, industry i has to produce: $\Delta x_i^{(1)} = \sum_{j=1}^{m} d_{ij}\,\Delta q_j^{(0)}$. Then, the additional production of industry i generates the need for intermediate goods, that is for commodity l: $\Delta q_l^{(1)} = \Delta u_{il}^{(1)} = a_{il}^{u}\,\Delta x_i^{(1)}$. The total intermediate demand for commodity l is: $\Delta q_l^{(1)} = \sum_{i=1}^{n} a_{il}^{u}\,\Delta x_i^{(1)}$. And the circuit is closed and begins again with this demand for commodity l. At step k, one has: $\Delta x_i^{(k)} = \sum_{j=1}^{m} d_{ij}\,\Delta q_j^{(k-1)}$, that is $\Delta\mathbf{x}^{(k)} = \mathbf{D}\,\Delta\mathbf{q}^{(k-1)}$, and $\Delta q_l^{(k)} = \sum_{i=1}^{n} a_{il}^{u}\,\Delta x_i^{(k)}$, that is $\Delta\mathbf{q}^{(k)} = \mathbf{A}^u\,\Delta\mathbf{x}^{(k)}$. Finally, $\Delta\mathbf{q}^{(k)} = \mathbf{A}^u\,\mathbf{D}\,\Delta\mathbf{q}^{(k-1)}$ and the model of (8) or (10) is recovered. In graphical terms, the circuit is as in figure 1:

Figure 1 about here

Obviously, the process could begin with demand made on an industry instead of demand for a commodity.

Remark. Kop Jansen and ten Raa (1990, p. 219) argue that the technology model is not scale invariant because when a sector grows more than others, its market share increases, which has an impact on technical coefficients without technical change. This argument is not false in itself, but it is based on a reverse functioning of the model, as advocated by Ghosh, by implicitly reversing **D**. The make matrix should be read as follows: a demand for commodity is served by an industry depending on its market share, that is column → row: matrix **D** cannot indicate how the growth of an industry generates an output of any commodity. The converse argument will apply in the next paragraph for the commodity-based technology.

With commodity-based technology, industries demand commodities by means of technical coefficients, but these commodities are assumed to be produced by industries in accordance with the industry output proportions, c_{ij}. If we are to translate this in terms of circuit, the process could begin with final demand for commodity *j*: $\Delta e_j^{(0)} \to \Delta q_j^{(0)} = \Delta e_j^{(0)}$. However, unlike **D**, the **C** matrix does **not** correctly indicate which industry will produce this commodity. If there are not the same number of industries and commodities, which is the general case, then obviously the inverse of C cannot be computed [5]. To understand what happens, the reader might consider the following rectangular example:

$$\mathbf{C} = \begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.1 & 0.6 & 0.3 \end{bmatrix} \begin{matrix} 1 \\ 1 \end{matrix} \quad \text{Industries}$$

$$\text{Commodities}$$

If you start by, say, $\Delta x_1 = 1000$ then you get: $\Delta q_1 = 0.5 \times 1000 = 500$, $\Delta q_2 = 0.3 \times 1000 = 300$ and $\Delta q_3 = 0.2 \times 1000 = 200$: you know how much the output of each commodity is increased. But if you start from a commodity (if **C** were square, you would compute the inverse of **C**) say, by $\Delta q_3 = 1000$, you cannot determine how much x_1 or x_2 will be increased. Put simply, you have no information in **C** to determine whether it is industry 1 or industry 2, or both, that will increase their output. Care is required with the meaning of the equals sign: here "=" does not mean that the right side "equals" the left side but that the left side implies the right side [6].

So, returning to the circuit, if it is a particular industry *i* -- arbitrarily chosen -- that produces this commodity *j*, that is $\Delta x_i^{(1)} = \frac{\Delta q_j^{(0)}}{c_{ij}}$, then *i* will have to increase its production of **all** the other commodities *l* that it usually produces, that is $\Delta q_l^{(1)} = c_{il}\, \Delta x_i^{(1)}$ to comply with the proportion described by **C**. It is a circuit, but a highly unrealistic one! If the process begins with demand $\Delta x_i^{(0)}$ made on an industry *i*, then this generates demand for intermediate commodities j by means of the technical coefficients a_{ji}^u: $\Delta q_j^{(1)} = a_{ji}^u\, \Delta x_i^{(0)}$; but after this, the above problem of allocation to the industries occurs again.

5 It is not a matter of computing pseudo-inverses or other artifices of computation.

6 In computer science, often the programming languages make this distinction between "=" ("equal") and ":=" ("put that value into this variable").

Figure 2 about here

Alternatively, it could be said that an industry *i*, producing commodities in accordance to coefficients c_{ij}, also demands intermediate commodities in accordance with the technical coefficients. So, demand $\Delta x_i^{(0)}$ made on an industry *i* implies that this industry increases its production of **all** commodities in accordance with $\Delta q_j^{(1)} = c_{ij}\,\Delta x_i^{(0)}$ and, simultaneously, that it demands some intermediate commodities in accordance with the technical coefficients: $\Delta q_l^{(1)} = a_{li}^{u}\,\Delta x_i^{(0)}$. But this is not a circular process because after this demand for intermediate commodities, there is no continuation.

Figure 3 about here

Even when the number of commodities miraculously equals the number of industries, the circuit is broken. Although it is mathematically true that $\mathbf{x} = \mathbf{C}'^{\,-1}\,\mathbf{q} \Leftrightarrow \mathbf{q} = \mathbf{C}'\,\mathbf{x}$, this does not mean that **x** is determined by **C**, i.e., $\Delta\mathbf{q} \rightarrow \Delta\mathbf{x} = \mathbf{C}'^{\,-1}\,\Delta\mathbf{q}$, only the contrary is true: $\Delta\mathbf{x} \rightarrow \Delta\mathbf{q} = \mathbf{C}'\,\Delta\mathbf{x}$. To explain this, consider the following square example:

$$\mathbf{C} = \underbrace{\begin{bmatrix} 0.5 & 0.3 & 0.2 \\ 0.1 & .06 & 0.3 \\ 0.1 & 0.2 & 0.7 \end{bmatrix}}_{\text{Commodities}} \begin{matrix} 1 \\ 1 \\ 1 \end{matrix} \quad \text{Industries}$$

While reading **C** by rows is correct (again, as row 1 is unchanged, $\Delta x_1 = 1000$ gives $\Delta q_1 = 500$, $\Delta q_2 = 300$ and $\Delta q_3 = 200$), reading **C** by columns is again false: after, say, $\Delta q_3 = 1000$, **C** could not indicate the value of Δx_1 and Δx_2; but computing the inverse of **C** is mathematically correct:

$$\mathbf{C}^{-1} = \underbrace{\begin{bmatrix} 2.25 & -1.0625 & -0.1875 \\ -0.25 & 2.0625 & -0.8125 \\ -0.25 & -0.4375 & 1.6875 \end{bmatrix}}_{\text{Industries}} \begin{matrix} 1 \\ 1 \\ 1 \end{matrix} \quad \text{Commodities}$$

Here (9) would indicate that, say, $\Delta q_3 = 1000$, generates $\Delta x_1 = -250$, $\Delta x_2 = -437.5$ and $\Delta x_3 = 1687.5$. However, this has no meaning in terms of circuit. The attentive reader will see also that negative terms are generated "from nothing" [7]. First, notice that if $|\mathbf{C}| \neq 0$, then:

(11) $$\mathbf{I} + (\mathbf{I} - \mathbf{C}) + (\mathbf{I} - \mathbf{C})^2 + (\mathbf{I} - \mathbf{C})^3 + \ldots(\mathbf{I} - \mathbf{C})^k \underset{k \to \infty}{\longrightarrow} \mathbf{C}^{-1}$$

So, the link between a coefficient of **C** and the corresponding coefficient of $\mathbf{C}^{-1}$ is very complicated and obviously does not correspond to the simple inverse of each individual

[7] As negative terms may always appear with commodity-based technology -- but not with industry-based technology -- for Kop Jansen and ten Raa (1990, p. 220) non-negativity is contradictory with the four axioms. So, I conclude that non negativity is itself an axiom.

coefficient. (11) also suggests an interpretation of the inverse of $\mathbf{C}$ in terms of internal circuit: $\mathbf{C}^{-1}$ corresponds to the sum of many terms $\mathbf{I}$, $\mathbf{I}-\mathbf{C}$, $(\mathbf{I}-\mathbf{C})^2$, etc., corresponding to many circuits, of length equal to respectively 0, 1 (from *i* to *j*), 2 (from *i* to *j* via any *l*), etc. What is the interpretation of such circuits, that are internal to matrix $\mathbf{C}$, without any role of $\mathbf{A}^u$. Who knows[8]! But they cannot be neglected because the coefficients c_{ij} have been defined first and not the coefficients of $\mathbf{C}^{-1}$: to compute the inverse of $\mathbf{C}$ is a valid matrix operation provided that $n = m$, but this is economically meaningless [9]. In the theory of valuated graphs, a negative coefficient corresponds to an arrow pointing in the reverse way. This is the case with many of the off-diagonal terms of $\mathbf{C}^{-1}$. So, returning to the circuit, the arrows $\{i, j\}$, correspond to a commodity *j* and an industry *i*, that have a negative coefficient must point in the reverse direction, not $j \to i$ (commodity $\to$ industry) but $i \to j$ (industry $\to$ commodity).

Figure 4 about here

Another interpretation of the input-output models lies in elementary probability theory (see the same analysis for the Leontief model in the appendix). A coefficient can be seen as a probability: the coefficient a_{ij}^u is the probability that industry *j* will spend one unit of money on commodity *i*, d_{ij} is the probability that a commodity *j* will be produced by industry *i* and c_{ij} is the probability that an industry *i* will produce a commodity *j*. As $\sum_i a_{ij}^u < 1$, there are leakages, but as $\sum_i a_{ij}^u + l_j = 1$, where $l_j = \frac{z_{ij}}{x_j}$, the terms a_{ij} and l_j are still probabilities.

So for the industry-based model, the matrix $\mathbf{A}^u\,\mathbf{D}$ is arrived at by the following reasoning. Commodity *j* has the probability d_{ij} of being produced by industry *i*, so when a quantity $q_j^{(k)}$ of commodity *j* is produced, the expectation of the output of industry *i* is $d_{ij}\,q_j^{(k)}$ of *j*. In total, for all commodities *j*, the expectation of the output of industry *i* is $E\left(x_i^{(k+1)}\right) = \sum_j d_{ij}\,q_j^{(k)}$ or $E\left(\mathbf{x}^{(k+1)}\right) = \mathbf{D}\,\mathbf{q}^{(k)}$. As the probability that industry *i* will buy an intermediate commodity *j* is a_{ji}^u, the expectation of the amount bought by industry *i* of commodity *j* is $a_{ji}^u\,E\left(x_i^{(k+1)}\right)$, that is the expectation of the amount bought by all industries *i* of commodity *j* is $E\left(q_j^{(k+1)}\right) = \sum_i a_{ji}^u\,E\left(x_i^{(k+1)}\right)$ or $E\left(\mathbf{q}^{(k+1)}\right) = \mathbf{A}^u\,E(\mathbf{x}^{(k+1)})$. Finally, $E\left(\mathbf{q}^{(k+1)}\right) = \mathbf{A}^u\,\mathbf{D}\,\mathbf{q}^{(k)}$ [10]. If it is

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8 However, brought to the fore by graph theory, circuits are well known in input-output economics where they are usually well-founded. For example, for **A**, the matrix of technical coefficients of the Leontief model, the quantity $(\mathbf{I} - \mathbf{A})^{-1} = \mathbf{I} + \mathbf{A} + \mathbf{A}^2 + \ldots$ can be interpreted as the sum of, respectively, the effect of final demand of any commodity *j* (say, cars), the direct effect of the intermediate demand between any couple of sectors *j* and *i* (say, cars that need steel), the indirect effect of the intermediate demand between any pairs of sectors *j* and *i* via any sector *l* (say, cars that need steel and steel that needs energy), etc.

9 It is not the case of $(\mathbf{I} - \mathbf{A})^{-1}$ in the Leontief model.

10 In other words, **D** is treated as a Markovian matrix of probability and $\mathbf{A}^u$ as a sub-Markovian matrix.

assumed that the true value of $\mathbf{q}^{(k+1)}$ tends to $E\left(\mathbf{q}^{(k+1)}\right)$, then the cycle resumes at the next step.

But for the square commodity-based model, such an interpretation in probabilistic terms is impossible: the probability that commodity j will be produced by industry i would be $(\mathbf{C}'^{\,-1})_{ij}$, the term $\{i, j\}$ of $\mathbf{C}'^{\,-1}$, but this "probability" may be negative: it is not a probability... and the computation as above is impossible [11]... So, computing the inverse of **C** is economically incorrect.

As can be seen, negative terms are not only a question of negative technical coefficients. These negative terms have long been misunderstood: many authors have tried to eliminate them or to test to see if they are due to errors of measurement (see ten Raa and van der Ploeg (1989), or Steenge (1990) with the introduction of a transition matrix between $\mathbf{A}^u$ and **C**), while they appear simply after a prohibited operation [12]. There is no need to worry then about negative terms: they will not arise!

IV. Commodity-based technology and push-process

In any case, this behavior corresponds to a **push-process**, similar to that assumed by Ghosh's hypothesis of a supply-driven model in traditional input-output economics. This appears to be in contradiction with the existence of technical coefficients in the Use matrix, i.e., a demand-driven hypothesis. However, to restore coherence, the perspective can be reversed by converting it into a full supply-driven model. Replace the technical coefficient matrix $\mathbf{A}^u$ by a matrix of allocation coefficients, $b^u_{ij} = \dfrac{u_{ij}}{q_i}$, that is:

(12) $$\mathbf{B}^u = \hat{\mathbf{q}}^{-1}\,\mathbf{U}$$

From (9), it follows:

(13) $$\mathbf{q} = \mathbf{C}'\,\mathbf{x}$$

From (12), we obtain

(14) $$\hat{\mathbf{q}}\,\mathbf{B}^u = \mathbf{U}$$

and substituting this in (2) gives $\mathbf{x} = \mathbf{B}^{u\prime}\,\hat{\mathbf{q}}\,\mathbf{s} + \mathbf{w} = \mathbf{B}^{u\prime}\,\mathbf{q} + \mathbf{w}$; so, the equation of the model is obtained by substituting (13) in this last equation:

(15) $$\mathbf{x} = \mathbf{B}^{u\prime}\,\mathbf{C}'\,\mathbf{x} + \mathbf{w}$$

which could be denoted $\mathbf{x} = \mathbf{B}'\,\mathbf{x} + \mathbf{w}$, with $\mathbf{B} = \mathbf{C}\,\mathbf{B}^u = \hat{\mathbf{x}}^{-1}\,\mathbf{V}\,\hat{\mathbf{q}}^{-1}\,\mathbf{U} = \langle \mathbf{V}\,\mathbf{s}\rangle^{-1}\,\mathbf{V}\,\langle \mathbf{V}'\,\mathbf{s}\rangle^{-1}\,\mathbf{U}$. This could be transformed into a commodity-commodity equation by premultiplying (15) by $\mathbf{C}'$ and using (4), that is: $\mathbf{q} = \mathbf{C}'\,\mathbf{B}^{u\prime}\,\mathbf{q} + \boldsymbol{\varpi}$, where $\boldsymbol{\varpi} = \mathbf{C}'\,\mathbf{w}$ is the value added by commodity, or $\mathbf{q} = \tilde{\mathbf{B}}'\,\mathbf{q} + \boldsymbol{\varpi}$, with $\tilde{\mathbf{B}} = \mathbf{B}^u\,\mathbf{C}$. Now, the push-process is constructed in complete conformity with a supply-driven model and it is circular. Supply of a commodity j generates an output from all

[11] In other words, even if **C** is a Markovian matrix of probability, $\mathbf{C}^{-1}$ is not.

[12] Ten Raa (1988) argues that negative terms are not due to errors in the data but to the model and he rejects the commodity-based model.

industries as indicated by $\mathbf{B}^u$, *à la* Ghosh, then the industries sell commodities in the proportions indicated by the coefficients c_{ij}. The initial increase $\Delta v_i^{(0)}$ in the value added of an industry i generates an equal increase in the output of this industry: $\Delta x_i^{(0)} = \Delta v_i^{(0)}$. By matrix $\mathbf{C}$, this generates an increase in the supply of all commodities: $\Delta q_j^{(1)} = c_{ij}\, \Delta x_i^{(0)}$, that is, all told, the increase in the supply of commodity j is: $\Delta q_j^{(1)} = \sum_{i=1}^{n} c_{ij}\, \Delta x_i^{(0)}$. This supplementary supply of a commodity j induces an increase in the output of all industries l following $\mathbf{B}^u$: $\Delta q_j^{(1)} \to \Delta x_l^{(1)} = b_{jl}^u\, \Delta q_j^{(1)}$, so, in total, industry l increases its output of $\Delta x_l^{(1)} = \sum_{j=1}^{m} b_{jl}^u\, \Delta q_j^{(1)}$. At step k, one has: $\Delta q_j^{(k)} = \sum_{i=1}^{n} c_{ij}\, \Delta x_i^{(k-1)}$ and $\Delta x_l^{(k)} = \sum_{j=1}^{m} b_{jl}^u\, \Delta q_j^{(k)}$, that is $\Delta \mathbf{q}^{(k)} = \mathbf{C}'\, \Delta \mathbf{x}^{(k-1)}$ and $\Delta \mathbf{x}^{(k)} = \mathbf{B}^{u\prime}\, \Delta \mathbf{q}^{(k)}$, so $\Delta \mathbf{x}^{(k)} = \mathbf{B}^{u\prime}\, \mathbf{C}'\, \Delta \mathbf{x}^{(k-1)}$. This is in conformity with the corresponding model (15): the supply-driven commodity-based-technology model is consistent [13].

Figure 5 about here

Note that this push model can be interpreted in probabilistic terms. The probability that an industry i will produce a commodity j is given by c_{ij}. So, when industry i produces a total output of $x_i^{(k)}$, the expectation that its output of commodity j will be $c_{ji}\, x_i^{(k)}$, that is for all industries, the expectation of quantity of commodity j produced overall is equal to $E\left(q_j^{(k+1)}\right) = \sum_i c_{ij}\, x_i^{(k)}$ or $E\left(\mathbf{q}^{(k+1)}\right) = \mathbf{C}'\, \mathbf{x}^{(k)}$. Then commodity j has a probability b_{ji} of being sold to industry i, that is the expectation of the quantity sold to industry i will be $b_{ji}^u\, q_j^{(k+1)}$, and in total, the expectation of industry i's purchases is $E\left(x_i^{(k+1)}\right) = \sum_j b_{ji}^u\, E\left(q_j^{(k+1)}\right)$ or $E\left(\mathbf{x}^{(k+1)}\right) = \mathbf{B}^{u\prime}\, E\left(\mathbf{q}^{(k+1)}\right)$. Finally $E\left(\mathbf{x}^{(k+1)}\right) = \mathbf{B}^{u\prime}\, \mathbf{C}'\, E\left(\mathbf{x}^{(k)}\right)$ and the cycle continues by assuming that the true value of $\mathbf{x}^{(k+1)}$ tends toward $E\left(\mathbf{x}^{(k+1)}\right)$.

There remains the question of compliance with the four axioms. However, a supply-driven model does not lead to the same formula for the material balance and the financial balance as the demand-driven model. The material balance corresponds to a resolution of the model by rows while the financial balance corresponds to a resolution of the model by columns. At equilibrium by rows, the Ghosh model must be $\mathbf{s} = \widetilde{\mathbf{B}}(\mathbf{U},\ \mathbf{V})\ \mathbf{s} + \mathbf{g}$ (where $\mathbf{g} = \hat{\mathbf{q}}^{-1}\ \mathbf{e}$), which can be found from (8), the material balance of the demand-driven model: this identity always holds because, substituting (14) in (3) gives $\mathbf{q} = \hat{\mathbf{q}}\ \mathbf{B}^u\ \mathbf{s} + \mathbf{e} \iff \hat{\mathbf{q}}\ \mathbf{s} = \hat{\mathbf{q}}\ \mathbf{B}^u\ \mathbf{s} + \mathbf{e} \iff \mathbf{s} = \mathbf{B}^u\ \mathbf{s} + \hat{\mathbf{q}}^{-1}\ \mathbf{e}$ $\iff \mathbf{s} = \mathbf{B}^u\ \mathbf{C}\ \mathbf{s} + \hat{\mathbf{q}}^{-1}\ \mathbf{e} \iff \mathbf{s} = \widetilde{\mathbf{B}}(\mathbf{U},\ \mathbf{V})\ \mathbf{s} + \mathbf{g}$. At equilibrium by columns, the supply-driven model is (15).

Price-invariance, that is, $\mathbf{B}(\hat{\mathbf{p}}\ \mathbf{U},\ \mathbf{V}\ \hat{\mathbf{p}}) = \hat{\mathbf{p}}\ \mathbf{B}(\mathbf{U},\ \mathbf{V})\ \hat{\mathbf{p}}^{-1}$ for all $\mathbf{p} > 0$ does not hold:

$$\mathbf{B}(\hat{\mathbf{p}}\ \mathbf{U},\ \mathbf{V}\ \hat{\mathbf{p}}) = \langle \mathbf{V}\ \hat{\mathbf{p}}\ \mathbf{s} \rangle^{-1}\ \mathbf{V}\ \hat{\mathbf{p}}\ \langle \hat{\mathbf{p}}\ \mathbf{V}'\ \mathbf{s} \rangle^{-1}\ \hat{\mathbf{p}}\ \mathbf{U} \neq \hat{\mathbf{p}}\ \langle \mathbf{V}\ \mathbf{s} \rangle^{-1}\ \mathbf{V}\ \langle \mathbf{V}'\ \mathbf{s} \rangle^{-1}\ \mathbf{U}\ \hat{\mathbf{p}}^{-1} = \hat{\mathbf{p}}\ \mathbf{B}(\mathbf{U},\ \mathbf{V})\ \hat{\mathbf{p}}^{-1}$$

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[13] All this is irrespective of the discussion about the artificial character of a supply-driven model: it is just to demonstrate that commodity-based technology is inconsistent, hesitating between a supply-driven and a demand-driven-model.

even if $\mathbf{B}^u$ does not depend on prices:

$$\mathbf{B}^u(\hat{\mathbf{p}}\,\mathbf{U},\ \mathbf{V}\,\hat{\mathbf{p}}) = \langle \hat{\mathbf{p}}\,\mathbf{V}'\,\mathbf{s}\rangle^{-1}\,\hat{\mathbf{p}}\,\mathbf{U} = \langle \mathbf{V}'\,\mathbf{s}\rangle^{-1}\,\hat{\mathbf{p}}^{-1}\,\hat{\mathbf{p}}\,\mathbf{U} = \langle \mathbf{V}'\,\mathbf{s}\rangle^{-1}\,\mathbf{U} = \mathbf{B}^u(\mathbf{U},\ \mathbf{V}).$$

Scale invariance, that is, $\mathbf{B}\left(\mathbf{U}\,\hat{\mathbf{k}},\ \hat{\mathbf{k}}\,\mathbf{V}\right) = \mathbf{B}(\mathbf{U},\ \mathbf{V})$ for all $\mathbf{k} > 0$ does not hold:

$$\mathbf{B}\left(\mathbf{U}\,\hat{\mathbf{k}},\ \hat{\mathbf{k}}\,\mathbf{V}\right) = \left\langle \hat{\mathbf{k}}\,\mathbf{V}\,\mathbf{s}\right\rangle^{-1}\,\hat{\mathbf{k}}\,\mathbf{V}\left\langle \left(\hat{\mathbf{k}}\,\mathbf{V}\right)'\,\mathbf{s}\right\rangle^{-1}\,\mathbf{U}\,\hat{\mathbf{k}} \neq \langle \mathbf{V}\,\mathbf{s}\rangle^{-1}\,\mathbf{V}\,\langle \mathbf{V}'\,\mathbf{s}\rangle^{-1}\,\mathbf{U} = \mathbf{B}(\mathbf{U},\ \mathbf{V})$$

even if $\mathbf{C}$ is scale invariant:

$$\mathbf{C}\left(\mathbf{U}\,\hat{\mathbf{k}},\ \hat{\mathbf{k}}\,\mathbf{V}\right) = \left\langle \hat{\mathbf{k}}\,\mathbf{V}\,\mathbf{s}\right\rangle^{-1}\,\hat{\mathbf{k}}\,\mathbf{V} = \langle \mathbf{V}\,\mathbf{s}\rangle^{-1}\,\hat{\mathbf{k}}^{-1}\,\hat{\mathbf{k}}\,\mathbf{V} = \langle \mathbf{V}\,\mathbf{s}\rangle^{-1}\,\mathbf{V} = \mathbf{C}(\mathbf{U},\ \mathbf{V}).$$

Like the demand-driven industry-based model, the supply-driven commodity-based model is not scale invariant, even if it is consistent in terms of circuit. However, the new model does not generate negative terms as the inverse of **C** is not computed.

V. Conclusion

Most national accounting systems are based on Stone's Make-Use model. Its two traditional alternative hypotheses have been explored. The first one, industry-based technology, is consistent in terms of circuit: its solution can be explained as a circuit, even in the rectangular case and it represents a fairly conventional demand-driven model. The alternative hypothesis, commodity-based technology, is problematic because the inverse of **C**, the matrix of industry output proportions, must be computed -- which is impossible in the rectangular case --. It is not well founded, generates inexplicable negative terms and suggests internal circuits inside **C** that lack credibility. Above all, $\mathbf{C}^{-1}$ fails to indicate how a commodity is distributed throughout industries. Consequently, the commodity-based model cannot be interpreted in terms of circuit: the problem of the negative terms generated in the direct requirement matrix is not simply an annoyance, but leads to rejection of the model. Moreover, this hypothesis means that the model vacillates between being demand-driven and supply-driven. However, if this demand-driven model is converted into a supply-driven one, as the commodity-based assumption no longer requires computation of an inverse matrix, the model recovers its coherence, even in the rectangular case, and can be interpreted as a circuit. To summarize, the industry-based technology corresponds to a demand-driven model while the commodity-based technology should be interpreted as a supply-driven model, which is a completely different thing.

Of course, it is annoying to be compelled to abandon one of Kop Jansen and ten Raa's two axioms, price-invariance and scale-invariance, either for the demand-driven industry-based model or for the supply-driven commodity-based model. However, as these two models do not generate negative terms, the question is one of choice between the three axioms and nonnegativity. It seems to be an axiomatic choice, but negativity implies so many drawbacks that scale-variance seems a minor evil in comparison: probably, one should prefer non-negativity, with the fatality of non-price-balance, price-variance and scale-variance [14]!

On the basis of this paper, the promoters of the System of National Accounts, United Nations and OECD, should take the opportunity of the introduction of new tables to reflect on the foundations of SNA, even if this calls into question long years of established practice. The industry-based model, that was adopted by the United States, should be carefully reconsidered: perhaps this with its internal circuit and its non negative terms, after abandoning price and scale invariance is not a too high price to pay for coherence...

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[14] Ten Raa's conclusion (1988, p. 535) is perhaps too pessimistic: even if negative terms are generated by the commodity-based model and even if the industry based model is not scale-invariant, the "very framework of deriving unique technical coefficients from the black box of a single pair of input and output flows must be abandoned". Put simply, the practitioner must know which set of axioms to choose.

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VII. Appendix: the circuit in the traditional Leontief model

Although long established, the traditional square model of input-output economics needs to be recalled here so that both its versions can be interpreted as a circuit. Denote x_j as the output of sector j, f_i as the final demand of commodity i, v_j as the value added of sector j; z_{ij} indicates how much of commodity i is bought by sector j, that is the flow from i to j:

$$\begin{matrix} \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix} & \begin{matrix} f_1 & x_1 \\ f_2 & x_2 \\ f_3 & x_3 \end{matrix} \\ \begin{matrix} v_1 & v_2 & v_3 \\ x_1 & x_2 & x_3 \end{matrix} & \end{matrix}$$

Matrix $\mathbf{Z}$ is homogenous by rows and columns. The central equation of traditional input-output economics (Leontief, 1936) is

(16) $$\mathbf{x} = \mathbf{A}\,\mathbf{x} + \mathbf{f}$$

where $a_{ij} = \frac{z_{ij}}{x_j}$ is the technical coefficient. The model can be resolved simply as:

(17) $$\mathbf{x} = (\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{f}$$

but it must also support reasoning based on as surge in demand, that is a differentiation of equation (16): $\Delta f_j^{(0)} \to \Delta x_j^{(0)} = \Delta f_j^{(0)} \to \Delta x_i^{(1)} = \Delta z_{ij}^{(1)} = a_{ij}\,\Delta x_j^{(0)}$. So, the total increase of the output of sector i is: $\Delta x_i^{(1)} = \sum_{j=1}^{n} a_{ij}\,\Delta x_j^{(0)}$. This continues at steps 2, ..., etc., and at step k: $\Delta x_i^{(k)} = \sum_{j=1}^{n} a_{ij}\,\Delta x_j^{(k-1)}$ that is $\Delta\mathbf{x}^{(k)} = \mathbf{A}\,\Delta\mathbf{x}^{(k-1)}$ and (16) is retrieved in derivative terms. The solution of the model is found by computing $\Delta\mathbf{x}^{(k)} = \mathbf{A}^k\,\Delta\mathbf{x}^{(0)} = \mathbf{A}^k\,\Delta\mathbf{f}$, thus the total increase of output is given by $\Delta\mathbf{x} = \sum_{k=1}^{n} \Delta\mathbf{x}^{(k)} = \left(\sum_{k=1}^{n} \mathbf{A}^k\right)\Delta\mathbf{f} \underset{k\to\infty}{\longrightarrow} (\mathbf{I} - \mathbf{A})^{-1}\,\Delta\mathbf{f}$; equation (17) is retrieved in derivative terms. The model is coherent. This is well known but it must not be overlooked: if the circuit solution is impossible, the ordinary solution cannot be interpreted and it is only an empty exercise and economically meaningless. Now, this second line of reasoning describes a circular process: production by a sector generates demand for some intermediate commodities described by the technical coefficients, which in turn generates production by the relevant sectors (remembering that the bijective sector-product correspondence is assumed). This model is described as *demand-driven*.

There is also an alternative *supply-driven* version of the model (Ghosh, 1958)[15]. Allocation coefficients $b_{ij} = \frac{z_{ij}}{x_i}$ are assumed to be stable. The central equation of the model is:

(18) $$\mathbf{x}'\,\mathbf{B} + \mathbf{v}' = \mathbf{x}'$$

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[15] It is often seen as less plausible (Bon, 1986; Oosterhaven, 1988, 1989, 1996; Miller, 1989; Gruver, 1989; Rose and Allison, 1989; and also: Dietzenbacher, 1997) but it must be recalled for the clarity of the exposé.

which solves as:

(19) $$\mathbf{x}' = \mathbf{v}' \, (\mathbf{I} - \mathbf{B})^{-1}$$

This could be also interpreted as a circuit. The initial increase $\Delta v_i^{(0)}$ of the value added of an industry i generates an equal increase in the output of this industry, $\Delta x_i^{(0)} = \Delta v_i^{(0)}$; this generates an increase in the supply of sector j: $\Delta x_j^{(1)} = b_{ij} \, \Delta x_i^{(0)}$. So the total increase in the output of sector j is $\Delta x_j^{(1)} = \sum_{i=1}^{n} b_{ij} \, \Delta x_i^{(0)}$, that is a step k: $\Delta x_j^{(k)} = \sum_{i=1}^{n} b_{ij} \, \Delta x_i^{(k-1)}$, or in matrix terms, $\Delta \mathbf{x}^{(k)\prime} = \Delta \mathbf{x}^{(k-1)\prime} \, \mathbf{B}$, and (18) is retrieved in derivative terms. The model solves as: $\Delta \mathbf{x}^{(k)\prime} = \Delta \mathbf{x}^{(0)\prime} \, \mathbf{B}^k = \Delta \mathbf{v}' \, \mathbf{B}^k$ and the increase in total output becomes $\Delta \mathbf{x}' = \sum_k \Delta \mathbf{x}^{(k)\prime} = \Delta \mathbf{v}' \left(\sum_k \mathbf{B}^k \right) = \Delta \mathbf{v}' \, (\mathbf{I} - \mathbf{B})^{-1}$: equation (19) is retrieved in derivative terms. The model is just as coherent as the demand-driven one. However, remember also that the two versions of the model are incompatible. As $\mathbf{B} = \hat{\mathbf{x}}^{-1} \, \mathbf{A} \, \hat{\mathbf{x}}$, allocation coefficients cannot be stable if technical coefficients are also stable: a star denoting aggregates after a change ($\mathbf{x}$ changing into $\mathbf{x}^*$), if $\mathbf{A}$ is stable, $\mathbf{A}^* = \mathbf{A}$, then $\mathbf{B}^* = \hat{\mathbf{x}}^{*\,-1} \, \mathbf{A} \, \hat{\mathbf{x}}^* \neq \mathbf{B}$.

Figure 6 about here

Figure 7 about here

Both models can be also interpreted in elementary probabilistic terms. In the demand-driven model, a_{ij} is the probability that j buys from i. So, for an output of $x_j^{(k)}$, the expectation of the amount bought by j to i is equal to $a_{ij} \, x_j^{(k)}$ and, overall, the expectation of what i sells is $E\left(x_i^{(k+1)}\right) = \sum_j a_{ij} \, x_j^{(k)}$, that is $E\left(\mathbf{x}^{(k+1)}\right) = \mathbf{A} \, \mathbf{x}^{(k)}$. For the supply-driven model, b_{ij} is the probability that i sells to j, and for an output of $x_i^{(k)}$, the expectation of what i sells to j is $b_{ij} \, x_i^{(k)}$; to the total, the expectation of what j buys equals $E\left(x_j^{(k+1)}\right) = \sum_i b_{ij} \, x_i^{(k)}$, or $E\left(\mathbf{x}^{(k+1)}\right) = \mathbf{B}' \, \mathbf{x}^{(k)}$. Both cycles begin again by assuming that the true value of $\mathbf{x}^{(k+1)}$ tends to $E\left(\mathbf{x}^{(k+1)}\right)$.

Figures *i*

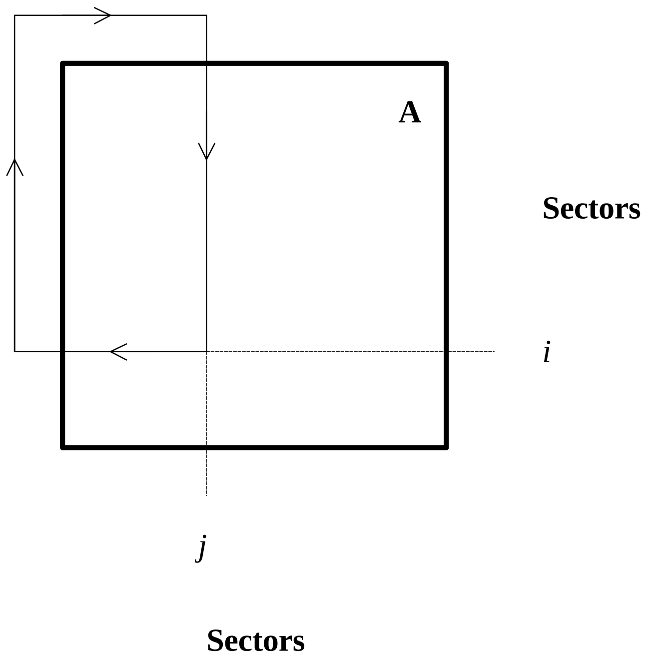

Figure 1. The circuit of the demand-driven Leontief model

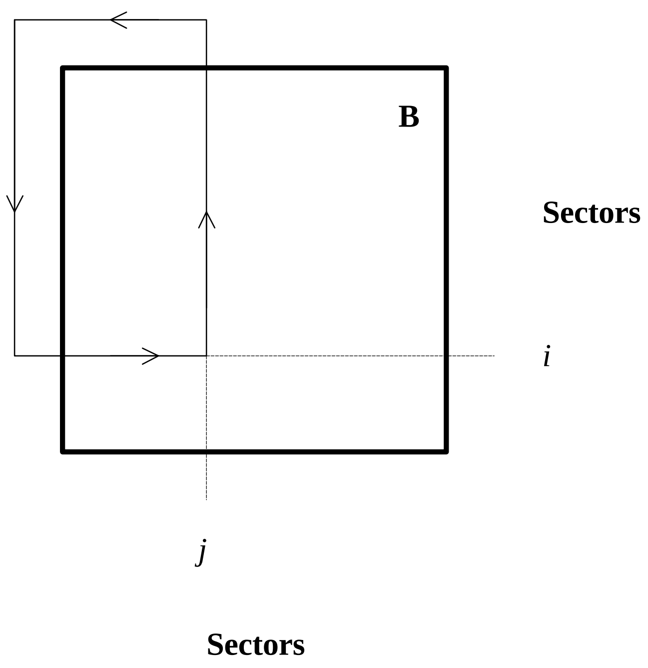

Figure 2. The circuit of the supply-driven Leontief model

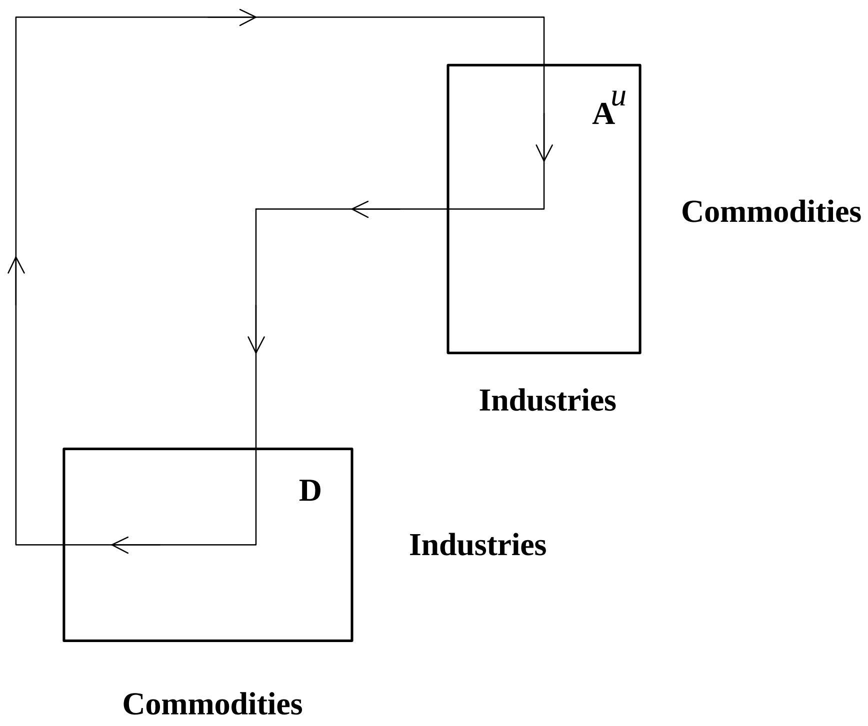

Figure 3. The circuit of the demand-driven industry-based model

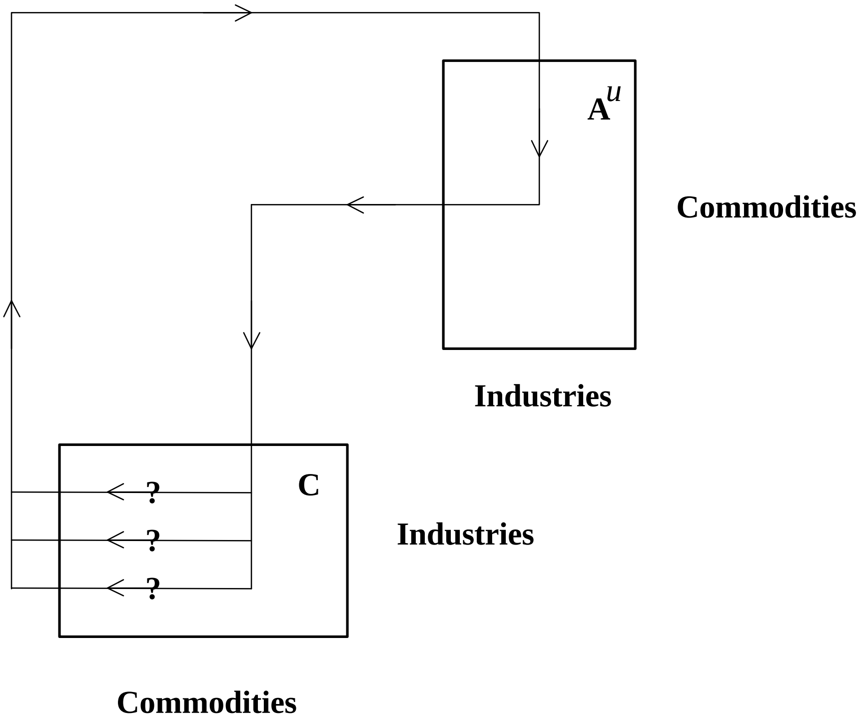

Figure 4. The undetermined circuits
of the rectangular demand-driven commodity-based model

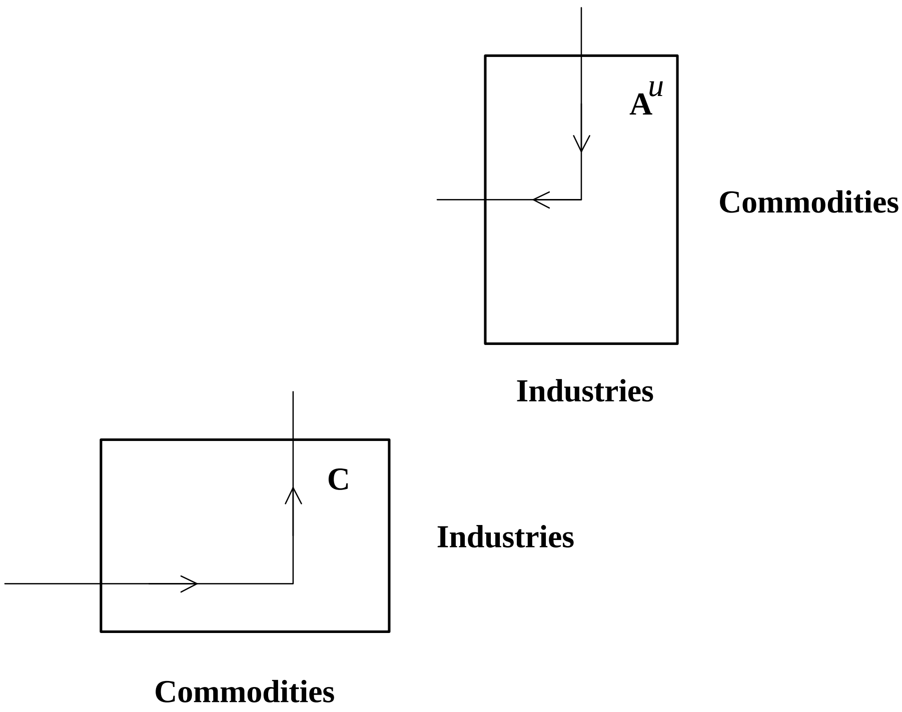

Figure 5. The broken circuit
of the rectangular demand-driven commodity-based model

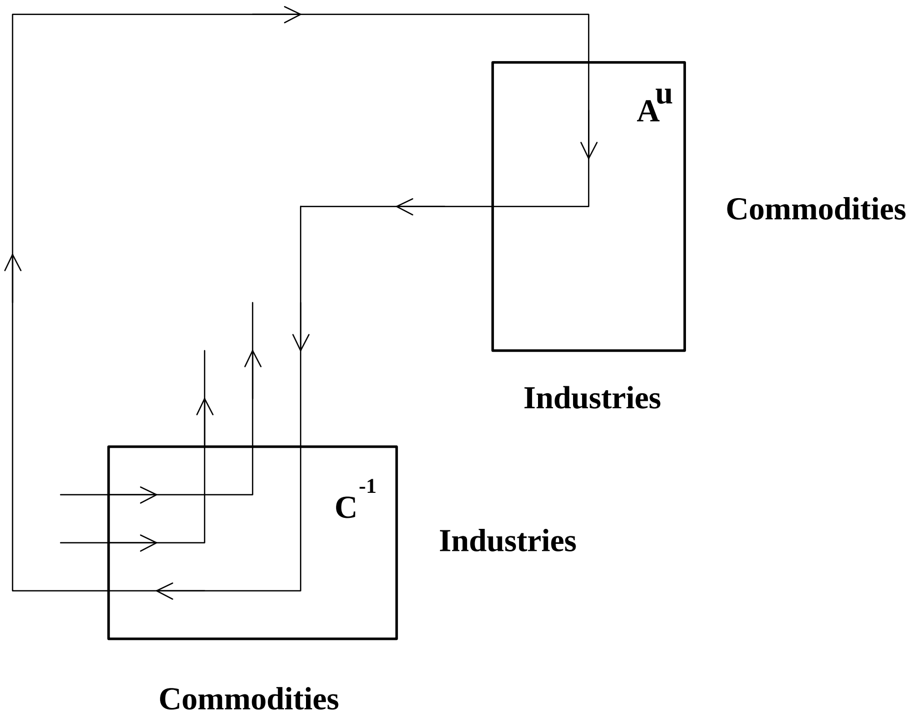

Figure 6. The reversed partial circuits
of the square demand-driven commodity-based model

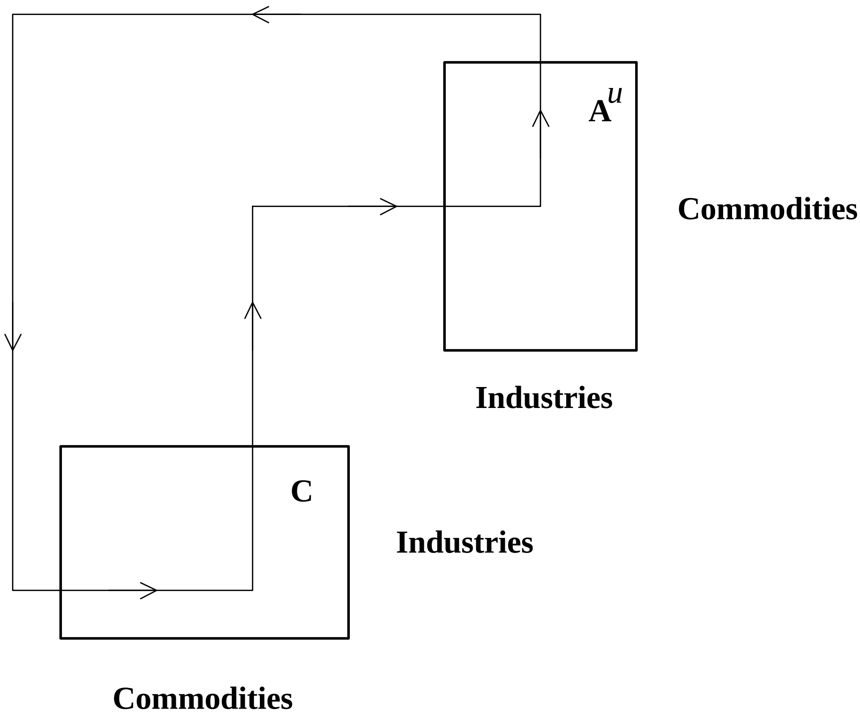

Figure 7. The circuit of the supply-driven commodity-based model