Failure of the normalization of the RAS method: absorption and fabrication effects are still incorrect

JEL classification. C63, C67, D57.

KEYWORDS. Input-Output, RAS, Biproportion.

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ABSTRACT. The **r** and **s** vectors of the RAS method of updating matrices are presented often as corresponding to an absorption effect and a fabrication effect. Here, it is proved that these vectors are not identified, so their interpretation in terms of fabrication and absorption effect is incorrect and even if a normalization was proposed to remove underidentification, this normalization fails and poses many difficulties.

I. Introduction

Stone and Brown (1962), followed by others, as Paelinck and Waelbroeck (1963), give an interpretation of the left and right multipliers, that are found when a RAS method is used to update a matrix, in terms of *absorption effect* and *fabrication effect* [1]. I begin by a presentation in conformity with Input-Output analysis but an extension apart of this field (transportation flows, demographic flows, etc.) will be presented.

Consider two matrices of technical coefficients, for two different dates $t = 0$ and $t = 1$, evaluated at the prices of $t = 1$: $\mathbf{A}^0$ and $\mathbf{A}^1$.Then the projection $\hat{\mathbf{A}}$ of $\mathbf{A}^0$ on the year $t = 1$ is the transformation of $\mathbf{A}^0$ such as this matrix gets the same margins than $\mathbf{A}^1$; this is given by: $\hat{\mathbf{A}} = RAS(\mathbf{A}^0,\ \mathbf{A}^1) = \mathbf{R}\ \mathbf{A}^0\ \mathbf{S}$, where $RAS(\ \ )$ is the operator well known as the "RAS method". In this case the diagonal matrix $\mathbf{R}$ is interpreted by the above authors as an absorption (or also substitution) effect: factors r_i affect a row of $\mathbf{A}^0$ and reflect the modification of the outlet of a commodity. The diagonal matrix $\mathbf{S}$ as a fabrication (or also transformation) effect: factors s_j affect a column of $\mathbf{A}^0$ and reflect the modification in the so-called "degree of fabrication". A similar interpretation can be done for other field than Input-Output analysis: when you consider transportation or demographic flows, for example, in a word any matrix of exchange, a matrix $\hat{\mathbf{Z}}$ is computed as $\hat{\mathbf{Z}} = \mathbf{R}\ \mathbf{Z}^0\ \mathbf{S}$ such as $\hat{\mathbf{Z}}$ has the same margins than another matrix $\mathbf{Z}^1$ and you have also a row effect for the left diagonal matrix $\mathbf{R}$ and a column effect for the right diagonal matrix $\mathbf{S}$ [2].

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1 This is exposed also by Snower (1990).

2 Note that, if $\mathbf{Z}^0$ is the flow matrix that corresponds to $\mathbf{A}^0$ and $\mathbf{A}^1$ to $\mathbf{Z}^1$, then it is demonstrated (Mesnard, 1994) that $RAS(\mathbf{A}^0, \mathbf{A}^1) = RAS(\mathbf{Z}^0, \mathbf{A}^1)$, but both are not necessarily

However, I will show that technical difficulties prevent them to be relevant of such interpretation.

II. Non identification of terms R and S

Even if this interpretation is commonly accepted, as the terms r_i and s_j are **not identified** [3], they **cannot** be interpreted for themselves as a absorption effect and a fabrication effect. If all the coefficients r_i are multiplied by λ then all the coefficients s_j are multiplied by $\frac{1}{\lambda}$ and conversely: multiplying fabrication effects by λ will divide absorption effects by λ, and conversely. This removes all signification to **R** and **S** in terms of fabrication or absorption effects. It is very simple to prove it. By commodity, the demonstration will be based on the algorithm presented by Bachem and Korte (1979) [4]:

(1) $$r_i = \frac{a^1_{i\bullet}}{\sum_{j=1}^{m} s_j\, a^0_{ij}} \text{ for all } i\text{, and } s_j = \frac{a^1_{\bullet j}}{\sum_{i=1}^{n} r_i\, a^0_{ij}} \text{ for all } j$$

This type of algorithm has an iterative numerical solution (k is an index of iteration), for example:

(2) $$r_i(k+1) = \frac{a^1_{i\bullet}}{\sum_{j=1}^{m} s_j(k)\, a^0_{ij}} \text{ for all } i\text{, and } s_j(k+1) = \frac{a^1_{\bullet j}}{\sum_{i=1}^{n} r_i(k+1)\, a^0_{ij}} \text{ for all } j$$

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equal to $RAS(\mathbf{Z}^0, \mathbf{Z}^1)$.

[3] I use this term by reference with econometrics, even if nothing is stochastic here.

[4] Bachem and Korte (1979) have proved that their algorithm is equivalent to the RAS algorithm.

After an initialization, for example by $s_j(0) = 1$, for all j, this leads to an equilibrium [5]:

(3) $$r_i^* = \frac{a_{i\bullet}^1}{\sum_{j=1}^{m} s_j^* \; a_{ij}^0} \text{ for all } i, \text{ and } s_j^* = \frac{a_{\bullet j}^1}{\sum_{i=1}^{n} r_i^* \; a_{ij}^0} \text{ for all } j$$

Now, assume that all $s_j(0)$ are replaced by $\tilde{s}_j(0) = \lambda\, s_j(0)$. Then, $r_i(1)$ will be replaced by $\tilde{r}_i(1)$:

(4) $$\tilde{r}_i(1) = \frac{a_{i\bullet}^1}{\sum_{j=1}^{m} \tilde{s}_j(0)\; a_{ij}^0} = \frac{a_{i\bullet}^1}{\sum_{j=1}^{m} \lambda\, s_j(0)\; a_{ij}^0} = \frac{1}{\lambda}\, r_i(1), \text{ for all } i$$

and

(5) $$\tilde{s}_j(1) = \frac{a_{\bullet j}^1}{\sum_{i=1}^{n} \tilde{r}_i(1)\; a_{ij}^0} = \frac{a_{\bullet j}^1}{\sum_{i=1}^{n} \frac{1}{\lambda}\, r_i(1)\; a_{ij}^0} = \lambda\, s_j(1), \text{ for all } j$$

By recurrence, r_i^* is replaced by $\tilde{r}_i^* = \frac{1}{\lambda}\, r_i^*$ and s_j^* by $\tilde{s}_j^* = \lambda\, s_j^*$. Similar results are found by reverting the role of the r and s terms. The same result holds for any exchange matrices, $\mathbf{Z}^0$ and $\mathbf{Z}^1$ instead of $\mathbf{A}^0$ and $\mathbf{A}^1$. So, r and s terms are not identified [6].

Remark. Note that the products $r_i\, s_j$, for all $(i,\, j)$, are identified, i.e., $\hat{r}_i^*\; \hat{s}_j^* = r_i^*\; s_j^*$ for all $(i,\, j)$, so it remains allowed to conduct a decomposition of change over time (Mesnard, 1990, 1997) (van der Linden and Dietzenbacher, 1995) because this one is based on the

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[5] It is not the aim of this paper to discuss about the properties of this equilibrium, and more generally, of biproportional algorithms; for further information, see (Bacharach, 1970), (Bachem and Korte, 1979), (Balinski and Demange, 1989), (Mesnard, 1994). Remember that it is demonstrated that all algorithms used to compute a biproportion, RAS or another, are equivalent and give the same result (Mesnard, 1994).

[6] A similar property can be found with other methods as soon as they are based on an iterative algorithm; for example, the bicausative method (Mesnard, 2000).

computation of the product $\mathbf{R}\,\mathbf{A}^0\,\mathbf{S}$ to be compared to $\mathbf{A}^1$, or $\mathbf{R}\,\mathbf{Z}^0\,\mathbf{S}$ to be compared to $\mathbf{Z}^1$, and not on the particular interpretation of **R** or **S**. ■

III. About a tentative of correction

Van der Linden and Dietzenbacher have proposed to normalize the substitution effect vector because the substitution effect is equal to zero for the whole economy (van der Linden and Dietzenbacher, 1995, p. 129, formula 13):

(6) $$\sum_i \sum_j r_i^* \, a_{ij}^0 \, s_j^* \, \frac{x_j^1}{\sum_j x_j^1} - \sum_i \sum_j a_{ij}^0 \, s_j^* \, \frac{x_j^1}{\sum_j x_j^1} = 0$$

where x_j^1 is the output of sector *j* at year 1; that is:

(7) $$\sum_i r_i^* \sum_j a_{ij}^0 \, s_j^* \, x_j^1 = \sum_i \sum_j a_{ij}^0 \, s_j^* \, x_j^1 \text{ or } \sum_i \sum_j \hat{a}_{ij} \, x_j^1 = \sum_i \sum_j a_{ij}^0 \, s_j^* \, x_j^1$$

The idea is interesting and clever because it seems to remove the unique level of underidentification, but some arguments can be introduce against it. First, this normalization is based on a certain interpretation of *r* terms: this interpretation is economic, not mathematical. If you choose another economic interpretation of the *r* terms, you have another formula and you cannot justify the normalization as it is.

Second, the normalization does not remove all degrees of freedom that cause non-identification. To understand it, you have to see that *r* and *s* terms are not independent, but linked by formula (1). For example, in formula (6) the terms s_j^* have to be replaced by their expression in formula (1), what gives a relation between the r_i^* only, but not linear: the terms r_i^* completely disappear from the right member of formula (6) what becomes:

(8) $$\sum_j a^1_{\bullet j}\, x^1_j = \sum_j a^1_{\bullet j}\, x^1_j \frac{\sum_i a^0_{ij}}{\sum_i a^0_{ij}\, r^*_i}$$

The left member of this expression simplifies as a constant, when the right member is hyperbolic (see the simple example in Annex 1). As it includes now the definition of the *r* terms, normally this expression would define exactly these terms (i.e., $r^*_i = \text{cst}$, for all i), but it is not the case (i.e., $r^*_n = \text{f}\,(r^*_1,\ r^*_2,\ ...,\ r^*_{n-1})$ for example): if the *r* terms are *n* in a *n*-dimension space, it leaves *n*-1 degrees of freedom. Normalization fails to fix completely the *r* terms and non-identification stays.

Remark. This is caused by the fact that there are more than one degree of non identification. When non-identification was exposed above, all *s* terms were assumed to be multiplied by the same constant λ, what implied that all the *r* terms were divided by the same constant λ. However, it is not the general case. In the general case, each s_j term could be multiplied by its own value of λ_j so there are *m* degrees of non-identification (or *n* degrees, if the initialization begins by the r_i), what increases the difficulty of the problem. ■

Third, even if it is admitted that the normalization of substitution effects leads to a solution and allows to remove the non-identification, it could bring some additional difficulties because if the global substitution effect is zero, the formula shows that some terms could be positive and some negative! When *r* or *s* terms are not positive, the interpretation of fabrication effects becomes difficult and you could obtain negative terms in the product matrix **R A S** when you accept negative terms inside matrices **R** or **S**, what cannot be correctly interpreted [7]. So,

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[7] Remember that the properties of convergence of the RAS method (Bacharach, 1970) (Mesnard 1994) are granted only if all terms of the **A** matrix are positive. If all *s* terms are positive, all *r* terms are positive, so if you choose all fabrication effects to be positive, you

positivity has to be added as two additional constraints: $r_i \geq 0$ for all *i* and $s_j^* \geq 0$ for all *j* (or even $r_i^* \in \Re^+$ and $s_j \in \Re^+$).

Four, the normalization condition does not take into account the fact that the equilibrium value of *r* and *s* terms must be found only iteratively: the condition has to be set not only for the formula at equilibrium but also for the formula at each step of the iterative computation of *r* and *s* terms:

(9) $$\sum_i r_i(k+1) \sum_j a_{ij}^0 \, s_j(k) \, x_j^1 = \sum_i \sum_j a_{ij}^0 \, s_j(k) \, x_j^1$$

where $s_j(k)$ can be replaced by $s_j(k) = \dfrac{a_{\bullet j}^1}{\sum_{i=1}^{n} r_i(k) \, a_{ij}^0}$, knowing that *r* and *s* terms are calculated by the iterative formula (2), under two additional constraints of positivity, $r_i(k) \geq 0$ for all *i* and *k*, and $s_j(k) \geq 0$ for all *j* and *k*:

(10) $$\sum_j a_{\bullet j}^1 \, x_j^1 \frac{\sum_i a_{ij}^0 \, r_i(k+1)}{\sum_{i=1}^{n} a_{ij}^0 \, r_i(k)} = \sum_j a_{\bullet j}^1 \, x_j^1 \frac{\sum_i a_{ij}^0}{\sum_{i=1}^{n} a_{ij}^0 \, r_i(k)}$$

The left member is no more a constant as in formula (8) and the result is not so simple because $r_i(k+1)$ is not equal to $r_i(k)$. Then, to be rigorous, one has to demonstrate that there is a path -- that respect the above constraints at each step *k* -- from the initialization to the equilibrium values of *r* and *s* terms [8]. Even when it is assumed for a particular matrix, **A**, that the solution exists, it has to be demonstrated that this solution can be reached without passing by some negative values of the terms $r_i(k)$ or $s_j(k)$, or it is necessary to impose the two additional constraints $r_i(k) \in \Re^+$ and $s_j(k) \in \Re^+$. Knowing an acceptable solution, it could seem

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have all substitution effects all positive also.

[8] This was demonstrated (Bacharach, 1970) in the case without on the *r* or *s* terms.

attractive to find a correct initialization of the process that corresponds to this rule. However, it is impossible to do this because the problem is transcendent [9].

<u>Remark</u>. In formual (10), if the terms $r_i(k)$ are considered by approximation as constant by respect to the terms $r_i(k+1)$, the above relation becomes linear at each step by respect to the $r_i(k+1)$.

Five, last but not least, the *r* and *s* terms are found as Lagrange multipliers of an optimization process as the minimization of the quantity of information (see Annex 2):

(11) $$\min \sum_i \sum_j \hat{a}_{ij} \log \frac{\hat{a}_{ij}}{a_{ij}^0},$$

under,

(12) $$\sum_j \hat{a}_{ij} = a_{i\bullet}^1 \text{ for all } i \text{ and } \sum_i \hat{a}_{ij} = a_{\bullet j}^1 \text{ for all } j.$$

The Lagrangian is: $L = \sum_j \sum_j \hat{a}_{ij} \log \frac{\hat{a}_{ij}}{a_{ij}^0} + \sum_i \lambda_i \left[\sum_j \hat{a}_{ij} - a_{i\bullet}^1 \right] + \sum_j \mu_j \left[\sum_i \hat{a}_{ij} - a_{\bullet j}^1 \right]$. Its maximization gives:

(13) $$\frac{\mathrm{d}\, L}{\mathrm{d}\, \hat{a}_{ij}} = 0 \Leftrightarrow \hat{a}_{ij} = \exp-(1+\lambda_i)\, a_{ij}^0 \exp-\mu_j, \text{ for all } i, j$$

Combined into the constraints, this gives:

(14) $$\exp-(1+\lambda_i) = \frac{a_{i\bullet}^1}{\sum_j a_{ij}^0 \exp-\mu_j} \text{ for all } i$$

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[9] There is a parallel with the computation of a general equilibrium: one has to demonstrate, knowing the initial conditions, that the equilibrium exists and that it can be reached by a feasible path, with no negative outputs.

(15) $$\exp{-\mu_j} = \frac{a^1_{\bullet j}}{\sum_i a^0_{ij}\ \exp{-(1+\lambda_j)}} \text{ for all } j$$

After a changing of variables, $r_i = \exp{-(1+\lambda_i)}$ for all i and $s_j = \exp{-\mu_j}$ for all j, this gives:

(16) $$\hat{a}_{ij} = r_i\, a^0_{ij}\, s_j$$

(17) $$r_i = a^1_{i\bullet}\left[\sum_j a^0_{ij}\, s_j\right]^{-1} \text{ for all } i$$

(18) $$s_j = a^1_{\bullet j}\left[\sum_i a^0_{ij}\, r_i\right]^{-1} \text{ for all } j$$

As the r and s terms are a simple transformation of the multipliers, the normalization appears to be a constraint on the multipliers, what is unusual and in contradiction with the concept of Lagrange multiplier (or Kuhn-Tucker multiplier).

IV. Conclusion

In the RAS method commonly used in Input-Output analysis, but also in other fields where exchange matrices are used (transportation flows, demographic flows, etc.), factors **R** and **S** have no signification at all by themselves because they are not identified. The interesting tentative of correction consisting into a normalization of substitution effects, following the argument that the total substitution effect is zero (van der Linden and Dietzenbacher, 1995), raises some important difficulties that prevent to consider it as acceptable. So, even if the RAS method and biproportional methods can be credited of some technical qualities [10] and even if RAS and biproportion can be seen as a simple generalization of "naïve" approaches (as the

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[10] As the non negativity of the projected matrix, or as the characteristic of the projected matrix to be the closer to the initial matrix in terms of information theory or entropy.

comparison of technical coefficient matrices), their economic interpretation remains difficult except if they are used to conduct an analysis of change over time.

V. Annexes

A. Annex 1

1. Computation of the constraints *r* terms at equilibrium

$$\mathbf{A}^0 = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.1 \end{bmatrix}, \mathbf{A}^1 = \begin{bmatrix} \; \end{bmatrix} \begin{matrix} 0.4 \\ 0.9 \end{matrix} \quad , x_1^1 = 20, x_2^1 = 25$$

$$0.5 \;\; 0.8$$

In the normalization expression,

$$r_1^* \; 0.1 \; s_1^* \; 20 + r_1^* \; 0.2 \; s_2^* \; 25 + r_2^* \; 0.3 \; s_1^* \; 20 + r_2^* \; 0.1 \; s_2^* \; 25$$

$$= 0.1 \; s_1^* \; 20 + 0.2 \; s_2^* \; 25 + 0.3 \; s_1^* \; 20 + 0.1 \; s_2^* \; 25$$

the terms $s_1^* = \dfrac{0.5}{0.1 \; r_1^* + 0.3 \; r_2^*}$ and $s_2^* = \dfrac{0.8}{0.2 \; r_1^* + 0.1 \; r_2^*}$ have to be inserted:

$$0.1 \; r_1^* \; \frac{0.5}{0.1 \; r_1^* + 0.3 \; r_2^*} \; 20 + 0.2 \; r_1^* \; \frac{0.8}{0.2 \; r_1^* + 0.1 \; r_2^*} \; 25$$

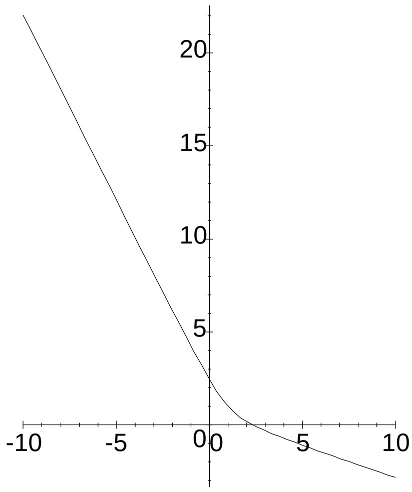
$$+ 0.3 \; r_2^* \; \frac{0.5}{0.1 \; r_1^* + 0.3 \; r_2^*} \; 20 + 0.1 \; r_2^* \; \frac{0.8}{0.2 \; r_1^* + 0.1 \; r_2^*} \; 25$$

$$= 0.1 \; \frac{0.5}{0.1 \; r_1^* + 0.3 \; r_2^*} \; 20 + 0.2 \; \frac{0.8}{0.2 \; r_1^* + 0.1 \; r_2^*} \; 25$$

$$+ 0.3 \; \frac{0.5}{0.1 \; r_1^* + 0.3 \; r_2^*} \; 20 + 0.1 \; \frac{0.8}{0.2 \; r_1^* + 0.1 \; r_2^*} \; 25$$

$$\Leftrightarrow 30 = \frac{4}{0.1 \; r_1^* + 0.3 \; r_2^*} + \frac{6}{0.2 \; r_1^* + 0.1 \; r_2^*}$$

$$r_2^* = 1.2222 - 1.1667\, r_1^* + 5.5556 \times 10^{-2} \sqrt{\left(484 - 420\, r_1^* + 225\, (r_1^*)^2\right)}$$



2. Iterative computation of the constraints *r* terms

The normalization expression is:

$$\left[0.1\, r_1(k+1) + 0.3\, r_2(k+1)\right] s_1(k)\, 20 + \left[0.2\, r_1(k+1) \ + 0.1\, r_2(k+1)\right] s_2(k)\, 25$$

$$= 0.4\, s_1(k)\, 20 + 0.3\, s_2(k)\, 25$$

The iterative terms are:

$$r_1(k+1) = \frac{0.4}{0.1\, s_1(k) + 0.2\, s_2(k)},\ r_2(k+1) = \frac{0.9}{0.3\, s_1(k) + 0.1\, s_2(k)}$$

$$s_1(k+1) = \frac{0.5}{0.1\, r_1(k+1) + 0.3\, r_2(k+1)},\ s_1(k+1) = \frac{0.8}{0.2\, r_1(k+1) + 0.1\, r_2(k+1)}$$

Replacing the terms s in the normalization expression gives:

$$10\,\frac{0.1\,r_1(k+1)+0.3\,r_2(k+1)}{0.1\,r_1(k)+0.3\,r_2(k)}+20\,\frac{0.2\,r_1(k+1)\;+0.1\,r_2(k+1)}{0.2\,r_1(k)+0.1\,r_2(k)}$$

$$=\frac{4}{0.1\,r_1(k)+0.3\,r_2(k)}+\frac{6}{0.2\,r_1(k)+0.1\,r_2(k)}$$

B. Annex 2

1. Remind about the computation of a biproportion

Consider two non-negative matrices $\mathbf{Z}^0$ and $\mathbf{Z}^1$. The result of a biproportion, $\hat{\mathbf{Z}} = K(\mathbf{Z}^0,\ \mathbf{Z}^1)$, is equal to $\mathbf{R}\,\mathbf{Z}^0\,\mathbf{S}$, where $\mathbf{R}$ and $\mathbf{S}$ are diagonal matrices. A biproportion must respect two conditions:

1) $\hat{\mathbf{Z}}$ must have the same row and column margins than $\mathbf{Z}^1$:

(19)
$$\left\{\begin{array}{l}\sum_j \hat{z}_{ij} = \sum_j z^1_{ij} \text{ for all } i \\ \sum_i \hat{z}_{ij} = \sum_i z^1_{ij} \text{ for all } j\end{array}\right\}$$

The $\mathbf{R}$ and $\mathbf{S}$ terms are there to guarantee the respect of this condition.

2) $\hat{\mathbf{Z}}$ is the matrix the nearest to $\mathbf{Z}^0$ following a certain criterion.

This criterion can be:

- The maximization of entropy (Wilson, 1970): max $-\sum_i \sum_j \hat{z}_{ij}\ \log \hat{z}_{ij}$, under the constraint $C = \sum_i \sum_j \hat{z}_{ij}\, c_{ij}$ where C is the total cost and c_{ij} is a cost, that can be considered as representative of $\mathbf{Z}^0$.

- Kullback and Liebler minimization of information (Kullback and Liebler, 1951), (Kullback, 1959), (Snickars and Weibull, 1977): min $\sum_i \sum_j \hat{z}_{ij} \log \frac{\hat{z}_{ij}}{z^0_{ij}}$.

- The minimization of interactions of Watanabe (1969) and Guiasu (1979), etc.

Stone's empirical method RAS respects the conditions. Historically, it was developed by Stone but the concept of biproportion was first formalized by Bacharach (1970).

The following algorithm is also correct (Bachem and Korte, 1979):

(20)
$$\left\{ \begin{array}{l} r_i = \dfrac{z^1_{i\bullet}}{\sum_{j=1}^{m} v_j \, z^0_{ij}} \text{ for all } i \\ s_j = \dfrac{z^1_{\bullet j}}{\sum_{i=1}^{n} u_i \, z^0_{ij}} \text{ for all } j \end{array} \right\}$$

Several algorithms respect the two conditions of a biproportion. And it is demonstrated (Mesnard, 1994) that all algorithms that respect the conditions of a biproportion lead necessarily to the same mathematical results [11]. Bacharach (1970) gives a demonstration of the uniqueness of the solution of RAS. As a biproportion can be deduced the minimization of an information function,

$$\min_{\hat{z}_{ij}} \ \mathrm{I} = \sum_i \sum_j \hat{z}_{ij} \log \frac{\hat{z}_{ij}}{z^0_{ij}} \text{ under } \sum_i \hat{z}_{ij} = z^1_{\bullet j} \text{ and } \sum_j \hat{z}_{ij} = z^1_{i\bullet}$$

and as this function, $\Re^n x \Re^m \rightarrow \Re$, of the $n \times m$ terms $\hat{z}_{ij}$ is continuous, derivable, on its compact interval of variation, the function –I is convex and it has a unique maximum on its interval of variation. Aït-Sahalia, Balinski and Demange (1988) have also established that the

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[11] Even if there can be some differences for computation speed and for the effect of successive rounds on the precision of computation, as studied by Bachem and Korte.

matrix that minimizes the information criteria is unique [12]. In conclusion, the computation of a biproportion is a "safe" operation, even it cannot be computed analytically.

2. Other methods to compute a biproportion

Also, there exists other methods to found a matrix $\hat{\mathbf{Z}}$, the nearest to a matrix $\mathbf{Z}^0$ that respects the margins of another matrix $\mathbf{Z}^1$ (i.e. under constraints of margins: $\sum_i \hat{z}_{ij} = \sum_i z_{ij}^1$ and $\sum_j \hat{z}_{ij} = \sum_j z_{ij}^1$):

- the minimization of the quadratic deviation (Frobenius norm of the difference matrix): $\min \sum_i \sum_j \left(\hat{z}_{ij} - z_{ij}^0\right)^2$,
- the minimization of the absolute differences: $\min \sum_i \sum_j \left|\hat{z}_{ij} - z_{ij}^0\right|$, what is not continuously derivable,
- the minimization of the Hölder norm at the power p: $\min \sum_i \sum_j \left|\hat{z}_{ij} - z_{ij}^0\right|^p$, knowing that the Hölder norm (Rotella and Borne, p. 78) is $\left\|\hat{\mathbf{Z}} - \mathbf{Z}\right\|_p = \left[\sum_i \sum_j \left|\hat{z}_{ij} - z_{ij}^0\right|^p\right]^{1/p}$, what is a generalization of two preceding,
- Pearson's χ^2: $\sum_i \sum_j \dfrac{\left(\hat{z}_{ij} - z_{ij}^0\right)^2}{z_{ij}^0}$,

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[12] Balinski and Demange (1989) have studied the axioms of biproportion in real numbers and in integers; see also (Aït-Sahalia, Balinski and Demange, 1988). This is applied to voting problems; see also Balinski and Young (1994) and Balinski and Gonzalez (1996). See also Toh (1998).

- Neyman's $\chi^2: \sum_i \sum_j \frac{\left(\hat{z}_{ij} - z_{ij}^0\right)^2}{\hat{z}_{ij}}$.

But generally these methods lead to various problems:

- non-linearities or non-differentiabilities in the found system as for Neyman or absolute differences,
- or negative terms in $\hat{\mathbf{Z}}$ as for the minimization of the Frobenius norm. Negative terms are impossible to explain in an input-output context: if $\mathbf{Z}^0$ has no negative terms, how to justify in an economic view point, the existence of some negative terms inside the projected matrix $\hat{\mathbf{Z}}$?

VI. Bibliographical references

AÏT-SAHALIA Yacine, Michel L. BALINSKI and Gabrielle DEMANGE, 1988. "Le redressement des tables de contingence: deux nouvelles approches", in *Mélanges Economiques*, *Essais en l'honneur de Edmond Malinvaud*, Economica - Editions EHESS.

BACHARACH, Michael, 1970. *Biproportional Matrices and Input-Output Change*. Cambridge University Press, Cambridge.

BACHEM, Achim and Bernhard KORTE, 1979. "On the RAS-Algorithm," *Computing* 23, 189-198.

BALINSKI Michel and Gabrielle DEMANGE, 1989. "An axiomatic approach to proportionality between matrices", *Mathematics of Operations Research*, Vol. 14, No 4, pp. 700-719.

BALINSKI Michel L. and H. P. YOUNG, 1994. "Apportionment", in S.M. POLLOCK and al., Eds., Handbooks in OR & MS, Vol. 6, Elsevier Science B.V.

BALINSKI Michel and Victoriano R. GONZALEZ, 1996. "Mexican Electoral Law", roneo, to appear in *Electoral Studies*.

GUIASU S., 1979. "An entropic measure of connection and interdependence between the subsystems of a given large system", *Proceedings the 3d formation symposium on mathematical methods for the analysis of large-scale systems (May 16-19, 1978),* Academia, Prague, pp. 113-124.

KULLBACK S., and R.A. LIEBLER, 1951. "On information and sufficiency", *Annals of Mathematics and Statistics*, Vol. 22, pp. 79-86.

KULLBACK S., 1959. *Information, Theory and Statistics*, Wiley.

van der LINDEN J. and E. DIETZENBACHER, 1995. "The nature of changes in the EU cost structure of production 1965-85: an RAS approach", in H.W. ARMSTRONG and R.W. VICKERMAN, Eds, *Convergence and divergence among European Regions*, Pion Limited, London, pp. 124-139.

de MESNARD, Louis, 1994. "Unicity of Biproportion", *SIAM Journal on Matrix Analysis and Applications* 2, 15, 490-495.

______ 1997. "A biproportional filter to compare technical and allocation coefficient variations", *Journal of Regional Science,* 37, 4, pp. 541-564.

______ 2000. "Bicausative matrices to measure structural change: are they a good tool?", to appear soon in *The Annals of Regional Science*.

PAELINCK, Jean and J. WAELBROECK, 1963. "Etude empirique sur l'évolution de coefficients «input-output». Essai d'application de la procédure «RAS» de Cambridge au tableau interindustriel belge", *Economie Appliquée*, No 6, pp. 82-111.

ROTELLA Frédéric and Pierre BORNE, 1995. *Théorie et pratique du calcul matriciel*, Editions Technip.

SNICKARS F., and J.W. WEIBULL, 1977. "A Minimum Information Principle. Theory and Practice.", *Regional Science and Urban Economics*, Vol. 7, pp. 137-168.

SNOWER, Dennis J., 1990. "New methods of updating input-output matrices", *Economic Systems Research*, 2, 1, pp. 27-37.

STONE R. and J.A.C. BROWN, 1962. "A long-term growth model for the British economy", in: R.C. GEARY (ed.), *Europe's future in figures*, North-Holland, Amsterdam.

TOH Mun-Heng, 1998. "The RAS Approach in Updating Input-Output Matrices: An Instrumental Variable Interpretation and Analysis of Structural Change", *Economic Systems Research* 10, 63-78.

WATANABE S., 1969. *Knowing and guessing. A quantitative study of inference and information*. Wiley, New-York.

WILSON A.G. 1970. *Entropy in Urban and Regional Modeling*, Pion.