

A biproportional measure of multidimensional inequality

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Keywords. Inequality, Biproportion, RAS, HET.

JEL Classification. C43, D63

Abstract. Bradburd and Ross have proposed a measure of multidimensional inequality based on a quadratic-loss criterion: one matrix is compared to another even if they have not the same margins. This is reconsidered. One removes the effect of size variation between the analyzed distribution and the reference distribution by giving to the two matrices the same margins with a biproportional operator. The size differences inside the analyzed structure are removed by a bimarkovian biproportional operator. As homogeneity does not signify equality, the homogeneous reference structure is replaced by a uniform matrix to be compared to the first matrix.

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(1635 words, this page excepted)

I. Introduction

Bradburd and Ross (1988) have introduced a measure of multidimensional inequality based on a quadratic loss criterion, that is a generalization of a measure introduced by Cowell (1977) for the one-dimensional case. Giving a two-dimensional variable X of which term $\{i, j\}$ is denoted x_{ij} , m_{ij} is the proportion of the element $\{i, j\}$ in the total, i.e., $m_{ij} = x_{ij}/x_{\bullet\bullet}$ where $x_{\bullet\bullet} = \sum_{i=1}^I \sum_{j=1}^J x_{ij}$, the multidimensional measure of inequality is:

$$(1) \quad S = \sum_{i=1}^I \sum_{j=1}^J I J \frac{(m_{ij} - \hat{m}_{ij})^2}{\hat{m}_{ij}}$$

where \hat{m}_{ij} is computed in the same manner than m_{ij} from a reference distribution \hat{X} .

This measure becomes simply a χ^2 if the reference distribution is the Bernoulli-Laplace distribution. It is referred as *HET* by Bradburd and Ross. It corresponds to a distribution \hat{M}^* that is proportional to the weight of each i and j , i.e., $m_{ij}^* = m_{i\bullet} m_{\bullet j}$ and the two distributions of i and j are independent:

$$(2) \quad HET = I J \sum_{i=1}^I \sum_{j=1}^J \frac{(m_{ij} - m_{i\bullet} m_{\bullet j})^2}{m_{i\bullet} m_{\bullet j}}$$

In this note, we discuss the implications of the S and *HET* measures and we propose some alternative measures.

Remark. If the industry is homogeneous, then $HET = 0$. This reminds one of Correspondence Analysis (Benzécri, 1992): in this factorial method, the χ^2 distance is used and the value of inertia is:

$$\Gamma = \frac{1}{x_{\bullet\bullet}} \sum_{i=1}^I \sum_{j=1}^J \frac{(x_{ij} - x_{ij}^*)^2}{x_{ij}^*} = \frac{1}{x_{\bullet\bullet}} \sum_{i=1}^I \sum_{j=1}^J \frac{\left(x_{ij} - \frac{x_{i\bullet} x_{\bullet j}}{x_{\bullet\bullet}}\right)^2}{\frac{x_{i\bullet} x_{\bullet j}}{x_{\bullet\bullet}}}$$

If the two variables are independent, $x_{ij} = \hat{x}_{ij}$ for all i and j and $\Gamma = 0$. ■

II. Removing the effect of size variations between the analyzed distribution and a reference distribution

In formula (1), when subtracting \hat{m}_{ij} from m_{ij} , one does not remove the effect of size variations of i categories and the size variations of j categories when one goes from M to \hat{M} (even if a global normalization is done when dividing x_{ij} by $x_{\bullet\bullet}$). In other words, $m_{i\bullet} \neq \hat{m}_{i\bullet}$ and $m_{\bullet j} \neq \hat{m}_{\bullet j}$ generally. If the marginal distributions of the two distributions M and \hat{M} are closed one with the other, this is not a big problem. However, if they differ largely this could be a difficulty because one could be interested in knowing what is the inequality between M and \hat{M} if the marginal distributions are identical, i.e., if the size differences are removed. In other words, this phenomenon, the size variations, has to be removed from the analysis of the differences between M and \hat{M} to obtain the pure differences between M and \hat{M} .

Remark. In the *HET* case, i.e., if \hat{M} corresponds to the Bernoulli-Laplace distribution that is in the χ^2 case, this argument falls because M and \hat{M} have the same marginal distributions: $m_{i\bullet} = \hat{m}_{i\bullet}$ and $m_{\bullet j} = \hat{m}_{\bullet j}$. ■

To take again the example of Bradburd and Ross (1988, p. 432), considering a perfectly homogeneous industry, $m_{i\bullet}$ is the proportion of industry shipments made by firm i (that is the market share of firm i), $m_{\bullet j}$ is the proportion of all industry shipments made in product category (or market) j and m_{ij} is the proportion of all industry shipments made by firm i in market j . So, if you compare directly m_{ij} and \hat{m}_{ij} , you confuse two phenomena: the size variations of industries i and the size variations of industries j from M to \hat{M} , in one hand, and the pure differences in the distribution of shipments from i to j , all things equal by elsewhere, in the other hand.

To summarize the above arguments, except in the *HET* case, it could be preferable to separate, in one hand, the effects of the variation of sizes (when one passes from M to \hat{M}) from

the pure effect of distribution, in a second hand. To perform this, the general idea consists into:

- Give to \mathbf{M} , the matrix corresponding to the distribution M , the same margins than $\hat{\mathbf{M}}$, the matrix corresponding to \hat{M} : this removes the effect of differences in size. This operation is performed by a biproportion, a generalization of the RAS method; about the idea of biproportion, the existence and the unicity of its solution, see Bacharach (1970) or Mesnard (1994). One have $K(\mathbf{M}, \hat{\mathbf{M}}) = \mathbf{U} \mathbf{M} \mathbf{V}$, where K is the operator of biproportion and \mathbf{U} and \mathbf{V} are diagonal matrices. For example, the following algorithm can be used (Bachem and Korte, 1979):

$$(3) \quad u_i = \frac{\hat{m}_{i\bullet}}{\sum_{j=1}^J v_j m_{ij}} \text{ for all } i, \text{ and } v_j = \frac{\hat{m}_{\bullet j}}{\sum_{i=1}^I u_i m_{ij}}$$

It is demonstrated that this algorithm will provide the same mathematical result (even if the computer efficiency may vary) than any other (Mesnard, 1994). So, one gives to \mathbf{M} the margins of $\hat{\mathbf{M}}$ by computing $K(\mathbf{M}, \hat{\mathbf{M}})$, as done in (Mesnard, 1990a, 1990b, 1996, 1997).

- Compare the resulting matrix to $\hat{\mathbf{M}}$ by computing:
 - a corrected S :

$$S = IJ \sum_{i=1}^I \sum_{j=1}^J \frac{\left[K(\mathbf{M}, \hat{\mathbf{M}})_{ij} - \hat{m}_{ij} \right]^2}{\hat{m}_{ij}}$$

- or, as done in (Mesnard, 1990a, 1990b, 1996, 1997):

$$V_A = \sqrt{\sum_{i=1}^I \sum_{j=1}^J \left[K(\mathbf{M}, \hat{\mathbf{M}})_{ij} - \hat{m}_{ij} \right]^2}$$

in absolute terms, and,

$$V_R = \frac{\sqrt{\sum_{i=1}^I \sum_{j=1}^J \left[K(\mathbf{M}, \hat{\mathbf{M}})_{ij} - \hat{m}_{ij} \right]^2}}{\sum_{i=1}^I \sum_{j=1}^J \hat{m}_{ij}}$$

in relative terms.

This method is referred to as the *ordinary biproportional filter* and has been applied to the dynamics of change in interindustrial relations (Mesnard, 1988, 1990a, 1990b, 1996) or to the determination of how is driven the economy, by the demand or by the supply (Mesnard 1997).

What about the *HET* measure? As said before, when $\hat{\mathbf{M}}$ is replaced by \mathbf{M}^* , the matrix that corresponds to the independence between the two marginal distributions, \mathbf{M} has the same margins than \mathbf{M}^* . So, the *HET* measure* is unchanged. First, $K(\mathbf{M}, \mathbf{M}^*) = \mathbf{M}$, because it is demonstrated in (Mesnard, 1994) that $K(\mathbf{P}, \mathbf{Q}) = \mathbf{P}$ for any matrices \mathbf{P} and \mathbf{Q} that have the same margins, i.e., $p_{i\bullet} = q_{i\bullet}$ for all i and $p_{\bullet j} = q_{\bullet j}$ for all j . Second, S becomes:

$$IJ \sum_{i=1}^I \sum_{j=1}^J \frac{\left[K(\mathbf{M}, \mathbf{M}^*)_{ij} - m_{ij}^* \right]^2}{m_{ij}^*} = IJ \sum_{i=1}^I \sum_{j=1}^J \frac{\left[m_{ij} - m_{ij}^* \right]^2}{m_{ij}^*}$$

what is equal to *HET*.

III. Removing the effect of size differences inside the analyzed distribution

The above biproportional method does not removes the effect caused by the size differences of individuals inside the analyzed distribution. This can be performed by giving to both \mathbf{M} and $\hat{\mathbf{M}}$ the same uniform margins, for example the margins of a bimarkovian matrix (Mesnard, 1998) by computing $K(\mathbf{M}, \mathbf{S})$ and $K(\hat{\mathbf{M}}, \mathbf{S})$, where \mathbf{B} is a (I, J) bimarkovian matrix with terms denoted s_{ij} (here, one is allowed to not take care to what is inside \mathbf{B}):

IV. Homogeneity and equality

With the *HET* measure, $\hat{\mathbf{M}}$ is computed as: $\hat{m}_{ij} = m_{i\bullet} m_{\bullet j}$. This is presented as corresponding to a homogeneous industry (Bradburd and Ross, 1988, p.432):

If industry shipments are homogeneous (i.e. each firm has the same proportion of its shipments in each product category j) then $m_{ij} = m_{i\bullet} m_{\bullet j}$, for all i and j . An industry is heterogeneous to the extent that it differs from this product for one or more firms.

This is a correct definition, but it is not sure that the criterion of homogeneity corresponds to the definition of equality, i.e., when the bidimensional distribution is heterogeneous, is it true to say that it is also unequal? It could be preferable to consider that the true egalitarian distribution is described by a bimarkovian matrix $\tilde{\mathbf{B}}$:

$$\tilde{\mathbf{B}} = \begin{bmatrix} 1 & \dots & 1 \\ \vdots & & \vdots \\ 1 & \dots & 1 \\ I & \dots & I \end{bmatrix} \begin{matrix} J \\ \dots \\ J \\ I \dots I \end{matrix}$$

In matrix $\tilde{\mathbf{B}}$, all industries have the same size (equality between industries), all markets have the same size (equality between markets), and all shipments are uniformly distributed (note that the intern structure of $\tilde{\mathbf{B}}$ plays a role now, but $\tilde{\mathbf{B}}$ and \mathbf{B} have the same margins).

So, one will compute $K(\mathbf{M}, \tilde{\mathbf{B}})$ and one obtain:

$$S = IJ \sum_{i=1}^I \sum_{j=1}^J \frac{\left[K(\mathbf{M}, \tilde{\mathbf{B}})_{ij} - \tilde{b}_{ij} \right]^2}{\tilde{b}_{ij}} = IJ \sum_{i=1}^I \sum_{j=1}^J \left[K(\mathbf{M}, \mathbf{B})_{ij} - 1 \right]^2$$

$$V_R = \frac{1}{IJ} \sqrt{\sum_{i=1}^I \sum_{j=1}^J \left[K(\mathbf{M}, \tilde{\mathbf{B}})_{ij} - \tilde{b}_{ij} \right]^2} = \frac{1}{IJ} \sqrt{\sum_{i=1}^I \sum_{j=1}^J \left[K(\mathbf{M}, \mathbf{B})_{ij} - 1 \right]^2}$$

knowing that, for any non negative matrix \mathbf{N} , one have $K(\mathbf{N}, \mathbf{N}) = \mathbf{N}$ (Mesnard, 1994).

V. Conclusion

The S measure of Bradburd and Ross (1988) allows to compare a bidimensional distribution to a bidimensional reference distribution by a quadratic-loss criterion: one matrix is compared to another even if they have not the same margins. This is reconsidered by introducing the biproportional measure of multidimensional inequality that allows to distinguish between two effects, an effect of size variation, between the analyzed structure and the reference structure, and a true effect of distribution. In addition, the bemarkovian biproportional measure of multidimensional inequality allows to remove the effect of size differences between individuals inside the analyzed distribution. In an axiomatic point of view, homogeneity is not equality, so the homogeneous reference structure used in the *HET* measure should be replaced by a uniform structure to be compared to the analyzed matrix.

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