Analyzing structural change: the biproportional mean filter and the biproportional bimarkovian filter

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ABSTRACT. The biproportional filter was created to analyze structural change between two input-output matrices by removing the effect of differential growth of sectors without predetermining if the model is demand or supply-driven, but with the disadvantage that projecting a first matrix on a second is not the same thing than projecting the second matrix on the first. Here two alternative methods are proposed which has not this last drawback, with the additional advantage for the biproportional bimarkovian filter that effects of sector size are also removed. Methods are compared with an application for France for 1980 and 1996.

I. Introduction

The basic idea inside biproportional methods of structural analysis is to become free from the hypothesis about orientation of the economy, demand driven or supply driven, by generalizing the comparison of two technical coefficient matrices, $\mathbf{A}$ and $\mathbf{A}^*$, as Leontief and followers did.

The choice made in (Mesnard, 1990a, 1990b, 1996, 1997) consist into indirectly generalize the simple comparison of two technical coefficient matrices [1] by working on two flow matrices $\mathbf{Z}$ and $\mathbf{Z}^*$, then giving to $\mathbf{Z}$ the same margins than $\mathbf{Z}^*$ by the mean of a biproportion, i.e. $K(\mathbf{Z},\ \mathbf{Z}^*)$, and then comparing $K(\mathbf{Z},\ \mathbf{Z}^*)$ to $\mathbf{Z}^*$. The justification of this method is the following: comparing two technical coefficient simply is the same thing than comparing two absolute values $z_{ij}\,\frac{x_j^*}{x_j}$ and z_{ij}^*:

$$a_{ij} \leftrightarrow a_{ij}^* \Leftrightarrow \frac{z_{ij}}{x_j} \leftrightarrow \frac{z_{ij}^*}{x_j^*} \Leftrightarrow z_{ij}\,\frac{x_j^*}{x_j} \leftrightarrow z_{ij}^*$$

where the symbol "↔" signifies: "compared to". So technical coefficient are not compared directly, but only indirectly.

So, in (Mesnard, 1990a, 1990b, 1996, 1977), a biproportional filter is proposed to analyze structural change. The principle consists into projecting the first flow table $\mathbf{Z}$ that one wants to compare to the margins of the second flow table $\mathbf{Z}^*$ to obtain $K(\mathbf{Z},\ \mathbf{Z}^*)$ and then to compare the result to the second table by calculating the difference matrix $\mathbf{Z}^* - K(\mathbf{Z},\ \mathbf{Z}^*)$. Here, this method is called the ordinary biproportional filter. See in annex some elements about biproportion.

The next step of the ordinary biproportional filtering method consists into computing the Frobenius norm

$$\sqrt{\sum_i \sum_j \left[z_{ij}^* - K(\mathbf{Z},\ \mathbf{Z}^*)_{ij} \right]^2}$$

of this difference matrix is calculated, or the Frobenius norm of vectors of this matrix, which is the absolute variation Σ between $\mathbf{Z}$ and $\mathbf{Z}^*$ for demanding sectors (column vectors, Σ_j) or supplying sectors (row vectors, Σ_i):

$$\Sigma_j = \sqrt{\sum_i \left[z_{ij}^* - K(\mathbf{Z},\ \mathbf{Z}^*)_{ij} \right]^2} \text{ and } \Sigma_i = \sqrt{\sum_j \left[z_{ij}^* - K(\mathbf{Z},\ \mathbf{Z}^*)_{ij} \right]^2}$$

Then, the relative variation is calculating by dividing the absolute variation by the total of the row or the column of flow matrix $\mathbf{Z}^*$.

$$\sigma_j = \frac{\sqrt{\sum_i \left[z_{ij}^* - K(\mathbf{Z},\ \mathbf{Z}^*)_{ij} \right]^2}}{\sum_i z_{ij}^*} \text{ and } \sigma_i = \frac{\sqrt{\sum_j \left[z_{ij}^* - K(\mathbf{Z},\ \mathbf{Z}^*)_{ij} \right]^2}}{\sum_j z_{ij}^*}$$

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[1] Remember that with simple coefficient matrices, there are two possibilities, either compare technical coefficient, either compare allocation coefficient matrices.

This remove the effect of change of both margins of the table from $\mathbf{Z}$ to $\mathbf{Z}^*$, i.e. the effect of differential growth of demanding and supplying sectors, without predetermining if the model is supply or demand driven.

Instead of projecting $\mathbf{Z}$ on the margins of $\mathbf{Z}^*$, what is called the *direct* calculation, different results can be also obtained when $\mathbf{Z}^*$ is projected to the margins of $\mathbf{Z}$ what is called the *reverse* calculation. Are calculated, $K(\mathbf{Z}^*,\ \mathbf{Z})$, then $\mathbf{Z} - K(\mathbf{Z}^*,\ \mathbf{Z})$ and,

$$\Sigma_j = \sqrt{\sum_i \left[z_{ij} - K(\mathbf{Z}^*,\ \mathbf{Z})_{ij} \right]^2}\ ,\ \Sigma_i = \sqrt{\sum_j \left[z_{ij} - K(\mathbf{Z}^*,\ \mathbf{Z})_{ij} \right]^2}$$

and,

$$\sigma_j = \frac{\sqrt{\sum_i \left[z_{ij} - K(\mathbf{Z}^*,\ \mathbf{Z})_{ij} \right]^2}}{\sum_i z_{ij}}\ ,\ \sigma_i = \frac{\sqrt{\sum_j \left[z_{ij} - K(\mathbf{Z}^*,\ \mathbf{Z})_{ij} \right]^2}}{\sum_j z_{ij}}$$

As these reverse results are not the same than direct results, it is necessary to conduct both computations (see Annex). This is a disadvantage, not only because it force to have two computations, but because results can diverge strongly and the ordering of sectors from the most changing to the less changing could be difficult as the application above will prove or as it can be seen in Mesnard (1990a, 1990b, 1996, 1997).

Fortunately, another choice could have been made to perform the analysis of structural change as a **direct** generalization of the simple coefficient comparison by generalizing the comparison of two technical coefficient matrices $\mathbf{A}$ and $\mathbf{A}^*$, i.e. two Markovian matrices. As before, this will not predetermine the orientation of the economy but without the disadvantage of two different results. I call this method the biproportional bimarkovian filter because it will based on the transformation of both matrices $\mathbf{Z}$ and $\mathbf{Z}^*$ into bimarkovian matrices by the mean of biproportion. The present article will develop this idea.

II. The problem

A. Basic idea

The basic idea of biproportional filtering consists into giving to the flow matrices $\mathbf{Z}$ and $\mathbf{Z}^*$ the same margins. The matrix $\mathbf{Z}^B$ which will provide these margins can be, for example, a third matrix of an intermediary year, 1988 if $\mathbf{Z}$ is 1980 and $\mathbf{Z}^*$ is 1996, but in fact only the margins of this matrix are important.

If $\mathbf{Z}^B$ has the same margins than $\mathbf{Z}$, or is equal to $\mathbf{Z}$, then $K(\mathbf{Z},\ \mathbf{Z}^B) = \mathbf{Z}$ and $K(\mathbf{Z}^*,\ \mathbf{Z}^B) = K(\mathbf{Z}^*,\ \mathbf{Z})$, so one have the reverse projection of the ordinary biproportional projector; if $\mathbf{Z}^B$ has the same margins than $\mathbf{Z}^*$, then $K(\mathbf{Z}^*,\ \mathbf{Z}^B) = \mathbf{Z}^*$ and $K(\mathbf{Z},\ \mathbf{Z}^B) = K(\mathbf{Z},\ \mathbf{Z}^*)$, so one have the direct projection of the ordinary biproportional projector. For all positions between these two "polar" matrices, one can obtain a wide range of results: I call this the *biproportional fixed-base filter*.

An good idea could consists into choosing $\mathbf{Z}^B$ such a manner that the variance would be maximized, either by measuring the variance in absolute terms as the square of the Frobenius norm of the difference matrix:

$$\|K(\mathbf{Z},\ \mathbf{Z}^B) - K(\mathbf{Z}^*,\ \mathbf{Z}^B)\|_F^2$$

either by measuring it in relative terms:

$$\frac{\|K(\mathbf{Z},\ \mathbf{Z}^B) - K(\mathbf{Z}^*,\ \mathbf{Z}^B)\|_F^2}{\sum_i \sum_j z_{ij}^B}$$

Unfortunately, these expressions are nonlinear regarding to the terms of $\mathbf{Z}$, $\mathbf{Z}^*$ and to the margins of $\mathbf{Z}^B$ and moreover they have no analytical solution because biproportion is a transcendent operator. So, such a problem can be solved only by a succession of computations of biproportion, what is too much heavy even for small matrices.

However, there are two particular matrices that are good candidates to play the role of $\mathbf{Z}^B$:

- a matrix, function of $\mathbf{Z}$ and $\mathbf{Z}^*$, for example the mean of $\mathbf{Z}$ and $\mathbf{Z}^*$, denoted $\bar{\mathbf{Z}}$ with $\bar{\mathbf{Z}} = \frac{1}{2}(\mathbf{Z} + \mathbf{Z}^*)$. I call this the *biproportional mean filter*.
- the bimarkovian matrix $\mathbf{1}^M$: this is the *biproportional bimarkovian filter*.

However, there is clear difference between these two methods: all column vectors have the same margin and all row vectors have the same margins in the biproportional bimarkovian filter, where it is not the case with the biproportional mean filter.

B. A representation by an Edgeworth box

A figure based on an Edgeworth box will illustrate it. Consider the matrices:

$$\mathbf{Z} = \begin{bmatrix} 5 & 5 \\ 4 & 1 \end{bmatrix} \begin{matrix} 10 \\ 5 \end{matrix} \quad \text{and } \mathbf{Z}^* = \begin{bmatrix} 3 & 1 \\ 6 & 5 \end{bmatrix} \begin{matrix} 4 \\ 11 \end{matrix}$$
$$\begin{matrix} 9 & 6 \end{matrix} \qquad\qquad \begin{matrix} 9 & 6 \end{matrix}$$

so,

$$K(\mathbf{Z},\ \mathbf{Z}^*) = \begin{bmatrix} 1.42 & 2.58 \\ 7.58 & 3.42 \end{bmatrix} \text{ and } K(\mathbf{Z}^*,\ \mathbf{Z}) = \begin{bmatrix} 6.74 & 3.26 \\ 2.26 & 2.74 \end{bmatrix}$$

This matrix is represented by the following Edgeworth box, where the sides of the box correspond to the column constraints of matrix $\mathbf{Z}$, the line *AB* corresponds to the row constraints of $\mathbf{Z}$ and where the point z corresponds to $\mathbf{Z}$; for matrix $\mathbf{Z}^*$, column constraints are the same and row constraints become the line *CD*, when $\mathbf{Z}^*$ is represented by point z^*. The length of segment $\{K(z,\ z^*),\ z^*\}$, which corresponds to the variation by the direct projection, is closed to the length of segment $\{K(z^*,\ z),\ z\}$, which corresponds to the variation by the reverse projection. Now, consider another matrix with the same margins than $\mathbf{Z}$:

$$\mathbf{Z}_1 = \begin{bmatrix} 8 & 2 \\ 1 & 4 \end{bmatrix} \begin{matrix} 10 \\ 5 \end{matrix}$$
$$\begin{matrix} 9 & 6 \end{matrix}$$

so,

$$K(\mathbf{Z}_1,\ \mathbf{Z}^*) = \begin{bmatrix} 3.74 & 0.26 \\ 5.26 & 5.74 \end{bmatrix}$$

As $\mathbf{Z}$ and $\mathbf{Z}_1$ have the same margins, $K(z^*, z_1)$ is confuse with $K(z^*, z)$. The segment $\{K(z_1, z^*), z^*\}$ is clearly shorter than the segment $\{K(z^*, z_1), z_1\}$. This is because the projection of z_1 is near the limit of the box (the orthogonal projection of z_1, found by an additive method, is even outside the limit of the box (it corresponds to negative terms in the projected matrix) and the ordinary biproportional projection corrects it.

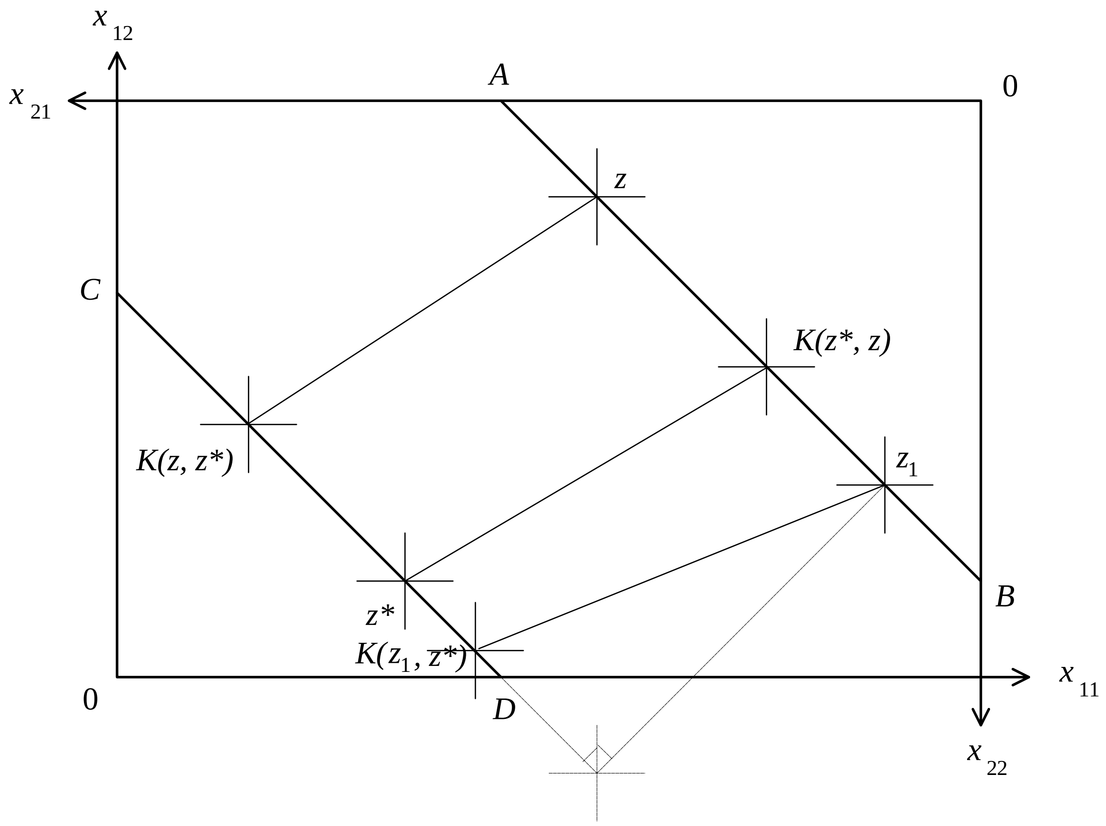

Figure 1. Edgeworth box for the ordinary biproportional projector

Consider the matrix:

$$\mathbf{Z}^B = \begin{bmatrix} . & . \\ . & . \end{bmatrix} \begin{matrix} 7.5 \\ 7.5 \end{matrix}$$
$$\quad 9 \quad 6$$

This matrix corresponds to the segment *EF* in the figure 2, but not to a precise point because the structure of the matrix is indifferent. This line can displaced between *AB* and *CD* and outside *AB* or *CD* and even by changing the size of the box. One have with the above $\mathbf{Z}^B$:

$$K(\mathbf{Z},\ \mathbf{Z}^B) = \begin{bmatrix} 3.31 & 4.19 \\ 5.69 & 1.81 \end{bmatrix},\ K(\mathbf{Z}^*,\ \mathbf{Z}^B) = \begin{bmatrix} 5.31 & 2.19 \\ 3.69 & 3.81 \end{bmatrix} \text{ and } K(\mathbf{Z}_1,\ \mathbf{Z}^B) = \begin{bmatrix} 6.61 & 0.89 \\ 2.39 & 5.11 \end{bmatrix}$$

Now, consider the matrix:

$$\bar{\mathbf{Z}} = \frac{1}{2}\,(\mathbf{Z}+\mathbf{Z}^{*}) = \begin{bmatrix} . & . \\ . & . \end{bmatrix} \begin{matrix} 7 \\ 8 \end{matrix}$$
$$\begin{matrix} 9 & 6 \end{matrix}$$

Then,

$$K\left(\mathbf{Z},\ \bar{\mathbf{Z}}\right) = \begin{bmatrix} 3.00 & 4.00 \\ 6.00 & 2.00 \end{bmatrix},\ K\left(\mathbf{Z}^{*},\ \bar{\mathbf{Z}}\right) = \begin{bmatrix} 5.00 & 2.00 \\ 4.00 & 4.00 \end{bmatrix} \text{ and } K\left(\mathbf{Z}_1,\ \bar{\mathbf{Z}}\right) = \begin{bmatrix} 6.25 & 0.75 \\ 2.75 & 5.25 \end{bmatrix}$$

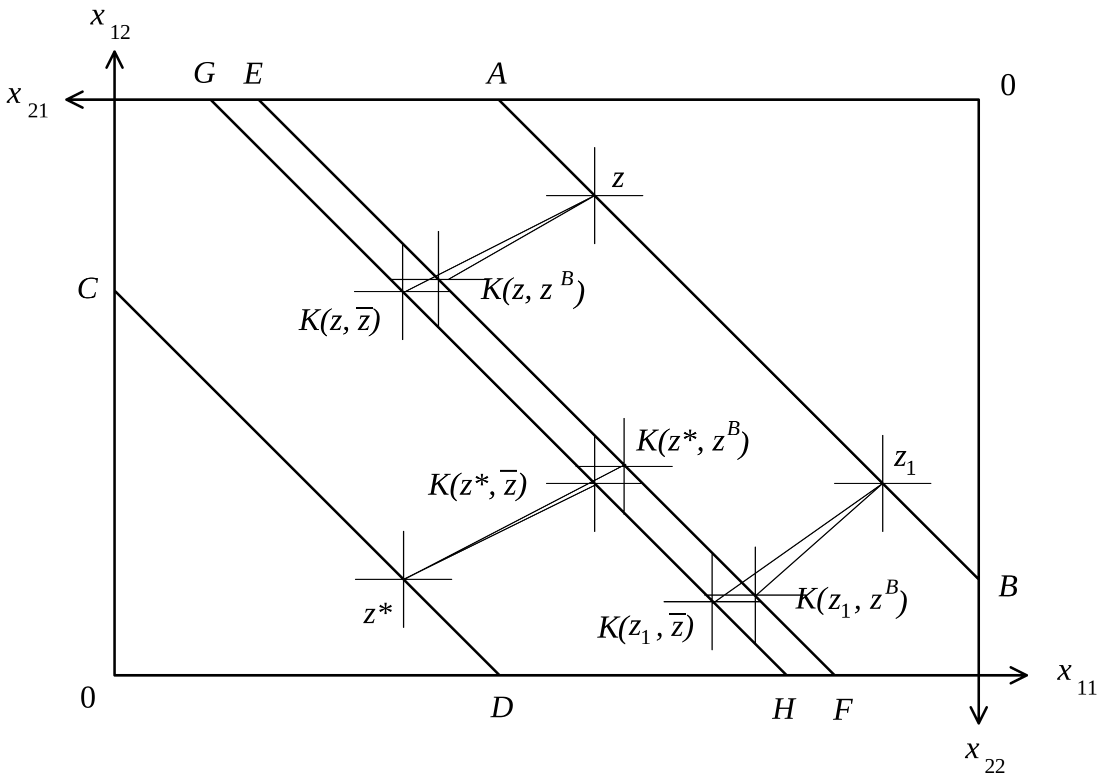

Figure 2. Edgeworth box for the single-base biproportional projector

A representation by an Edgeworth box is not possible for the biproportional bimarkovian filter [2] because the size of the box changes completely.

III. The methods

A. The biproportional single-base filter and the biproportional mean filter

1. First step

Each matrix $\mathbf{Z}$ and $\mathbf{Z}^{*}$ is projected to the margins of another intermediary matrix $\mathbf{Z}^{B}$ which provide a fixed base, for example the mean $\bar{\mathbf{Z}}$ of $\mathbf{Z}$ and $\mathbf{Z}^{*}$, to give $K(\mathbf{Z},\ \mathbf{Z}^{B})$ and $K(\mathbf{Z}^{*},\ \mathbf{Z}^{B})$

[2] Also, note that for these school-case (2, 2) matrices, the corresponding bimarkovised matrices are always symmetrical, what removes much interest in this example for the biproportional bimarkovian filter. At least, matrices must be (3, 3).

with the biproportional mean filter. This operation allows to remove the effects of differential growth of sector.

There are many tools to perform this operation of projection of a matrix and the problem is to choose one of these tools, or, in other words, there are an infinite number of matrices that can have the same margins and the problem is to choose one of these matrices. The resulting matrix may vary depending on the tool chosen to perform the projection, and consequently the results of the methods may vary (see annex). Here, we choose biproportion. In particular, with some methods like the minimization of the least squares between a matrix, $\mathbf{Z}$ or $\mathbf{Z}^*$, and the matrix $\mathbf{Z}^B$, this may create negative terms in $K(\mathbf{Z},\ \mathbf{Z}^B)$ because the form becomes additive: $K(\mathbf{Z},\ \mathbf{Z}^B) = \mathbf{U} + \mathbf{Z} + \mathbf{V}$, where $\mathbf{U}$ and $\mathbf{V}$ are diagonal matrices. This is why the tool that we choose to perform binormalization is biproportion. With biproportion, one have $K(\mathbf{Z},\ \mathbf{Z}^B) = \mathbf{U}\ \mathbf{Z}\ \mathbf{V}$, and,

$$u_i = \frac{z_{i\bullet}^B}{\sum_{j=1}^{m} v_j\ z_{ij}}\ \text{, for all } i \text{ , and } v_j = \frac{n}{m}\ \frac{z_{\bullet j}^B}{\sum_{i=1}^{n} u_i\ z_{ij}}\text{, for all } j$$

so, if all terms $u_i^{(k)}$ are positive, all terms $v_j^{(k)}$ will be also positive, as soon as all terms of $\mathbf{Z}$ are positive (Mesnard, 1994, 1997). The same reasoning holds for $\mathbf{Z}^*$. One have the guarantee that there will be no negative terms inside the projected matrix if there are no inside $\mathbf{Z}$ or $\mathbf{Z}^*$. So, the operation of projection by means of a biproportional method signifies that the projected matrix is the nearest to the initial matrix, in the sense of information theory among other theories, see (Mesnard, 1990a), but with the guarantee that there will be no negative terms inside the projected matrix if there are no inside Z or $\mathbf{Z}^*$.

2. Second step

$K(\mathbf{Z},\ \mathbf{Z}^B)$ is compared to $K(\mathbf{Z}^*,\ \mathbf{Z}^B)$ by calculating the Frobenius norm of the difference matrix $K(\mathbf{Z}^*,\ \mathbf{Z}^B) - K(\mathbf{Z},\ \mathbf{Z}^B)$. This is done in absolute values,

- for one single coefficient Σ_{ij}:

$$\Sigma_{ij} = \left(K(\mathbf{Z}^*,\ \mathbf{Z}^B)_{ij} - K(\mathbf{Z}^*,\ \mathbf{Z}^B)_{ij} \right)^2$$

- for demanding sectors (i.e. for column vectors, Σ_j):

$$\Sigma_j = \sqrt{\sum_i \left(K(\mathbf{Z}^*,\ \mathbf{Z}^B)_{ij} - K(\mathbf{Z}^*,\ \mathbf{Z}^B)_{ij} \right)^2}$$

- for supplying sectors ((i.e. for row vectors, Σ_i):

$$\Sigma_i = \sqrt{\sum_j \left(K(\mathbf{Z}^*,\ \mathbf{Z}^B)_{ij} - K(\mathbf{Z}^*,\ \mathbf{Z}^B)_{ij} \right)^2}$$

- or for the whole economy, Σ:

$$\Sigma = \sqrt{\sum_i \sum_j \left(K(\mathbf{Z}^*,\ \mathbf{Z}^B)_{ij} - K(\mathbf{Z}^*,\ \mathbf{Z}^B)_{ij} \right)^2}$$

This is done also in relative terms, by dividing absolute variabilities by the value of the margin of $\mathbf{Z}^B$,

- for one single coefficient, σ_{ij}:

$$\sigma_{ij} = \frac{\left(K(\mathbf{Z}^*,\ \mathbf{Z}^B)_{ij} - K(\mathbf{Z}^*,\ \mathbf{Z}^B)_{ij}\right)^2}{z^B_{ij}}$$

- for demanding sectors ((i.e. for column vectors, σ_j):

$$\sigma_j = \frac{\sqrt{\sum_i \left(K(\mathbf{Z}^*,\ \mathbf{Z}^B)_{ij} - K(\mathbf{Z}^*,\ \mathbf{Z}^B)_{ij}\right)^2}}{z^B_{\bullet j}}$$

- for supplying sectors ((i.e. for row vectors, σ_i):

$$\sigma_i = \frac{\sqrt{\sum_j \left(K(\mathbf{Z}^*,\ \mathbf{Z}^B)_{ij} - K(\mathbf{Z}^*,\ \mathbf{Z}^B)_{ij}\right)^2}}{z^B_{i\bullet}}$$

- or for the whole economy, σ:

$$\sigma = \frac{\sqrt{\sum_i \sum_j \left(K(\mathbf{Z}^*,\ \mathbf{Z}^B)_{ij} - K(\mathbf{Z}^*,\ \mathbf{Z}^B)_{ij}\right)^2}}{z^B_{\bullet\bullet}}$$

Remark. Because of their not linear nature, relative variabilities have not the property of aggregation. So, one can compare relative variabilities for sectors between them, or relative variabilities for individual cells between them, or even relative variabilities for the whole economy at different dates, but must not compare the relative variability for the whole economy to the relative variabilities for sectors, or relative variabilities for sectors to relative variabilities for individual cells.

For the biproportional mean filter, matrix $\mathbf{Z}^B$ is simply replaced by matrix $\bar{\mathbf{Z}}$.

B. Particular case of the biproportional bimarkovian filter

In the biproportional bimarkovian filter, the matrix $\mathbf{Z}^B$ becomes the bimarkovian matrix $\mathbf{1}^M$. Not only the effect of the differential growth of sectors will be removed without predetermining if the economy is demand or supply- driven as in the ordinary biproportional filter, not only the problem of the double result will be removed as in the biproportional single-base or the biproportional mean filter, but as an additional advantage of the biproportional bimarkovian filter, the effect of differential size of sectors will be removed: after projection all sectors in column will have the same margin, i.e. the same size, and all sectors in column will have the same margin.

1. First step

Each matrix $\mathbf{Z}$ and $\mathbf{Z}^*$ is transformed into a bimarkovian matrix, $\mathbf{Z}^M$ and $\mathbf{Z}^{*M}$. A bimarkovian or binormalized matrix is a matrix of which all margins in a same side, column or row, are equal to 1 [3]. In fact, this is exactly possible only for square matrices. For

[3] In fact, any other number can be chosen, the important thing is that all margins of the same side would be equal.

rectangular matrices or dimension $(n,\ m)$, the margins of one side, say the side of dimension n, are equal to μ, and the margins of the other side are equal to λ:

$\mathbf{1}^M\ \mathbf{s} = \mu\ \mathbf{s}$ and $\mathbf{s}'\ \mathbf{1}^M = \lambda\ \mathbf{s}'$

For example, one can take:

(1) $$\mathbf{1}^M = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{matrix} 1 \\ \dots \\ 1 \end{matrix} \qquad \begin{matrix} \frac{n}{m} & \dots & \frac{n}{m} \end{matrix}$$

(2) $$\mathbf{1}^M = \begin{bmatrix} & & \\ & & \\ & & \end{bmatrix} \begin{matrix} m \\ \dots \\ m \end{matrix} \qquad \begin{matrix} n & \dots & n \end{matrix}$$

This operation of projection with the biproportional bimarkovian filter not only removes the effects of differential growth of sectors, but also it removes the effect of difference of size of sectors.

As said before, there are many tools to project a matrix and the results may vary depending on the tool chosen to perform it: here biproportion is chosen: the name of the filter is not the *bimarkovian filter* but the *biproportional bimarkovian filter*.

In particular, with some methods like the minimization of the least squares between a matrix, $\mathbf{Z}$ or $\mathbf{Z}^*$, and the matrix $\mathbf{1}^M$, this may create negative terms in $K(\mathbf{Z},\ \mathbf{1}^M)$ because the form becomes additive: $K(\mathbf{Z},\ \mathbf{1}^M) = \mathbf{U} + \mathbf{Z} + \mathbf{V}$, where $\mathbf{U}$ and $\mathbf{V}$ are diagonal matrices. This is why the tool that we choose to perform binormalization is biproportion. With biproportion, one have $K(\mathbf{Z},\ \mathbf{1}^M) = \mathbf{U}\ \mathbf{Z}\ \mathbf{V}$, and,

- with matrix $\mathbf{1}^M$ defined in (1):

(3) $$u_i = \frac{1}{\sum_{j=1}^{m} v_j\ z_{ij}}\ \text{, for all } i\text{, and } v_j = \frac{n}{m}\ \frac{1}{\sum_{i=1}^{n} u_i\ z_{ij}}\text{, for all } j$$

- or with the matrix $\mathbf{1}^M$ defined in (2):

(4) $$u_i = \frac{m}{\sum_{j=1}^{m} v_j\ z_{ij}}\text{, for all } i\text{, and } v_j = \frac{n}{\sum_{i=1}^{n} u_i\ z_{ij}}\text{, for all } j$$

so, if all terms $u_i^{(k)}$ are positive, all terms $v_j^{(k)}$ will be also positive, as soon as all terms of $\mathbf{Z}$ are positive (Mesnard, 1994, 1997). The same reasoning holds for $\mathbf{Z}^*$. One have the guarantee that there will be no negative terms inside the projected matrix if there are no inside $\mathbf{Z}$ or $\mathbf{Z}^*$. Effectively, a separable modification of $\mathbf{Z}$ (or $\mathbf{Z}^*$) is ineffective (Mesnard, 1994, 1997): if $\mathbf{Z}$ is replaced by $\tilde{\mathbf{Z}} = \mathbf{\Psi}\ \mathbf{Z}\ \mathbf{\Omega}$, then $K\left(\tilde{\mathbf{Z}},\ \mathbf{1}^M\right) = K(\mathbf{Z},\ \mathbf{1}^M)$, and the exact form of the bimarkovian matrix has no importance. So, the above expressions (3) and (4) are equivalent.

2. Second step

$\mathbf{Z}^M$ is compared to $\mathbf{Z}^{*M}$, i.e. the Frobenius norm of the difference matrix $\mathbf{Z}^{*M} - \mathbf{Z}^M$ is calculated. The rest of the method is similar as for the biproportional single-base filter, except that absolute variations and relative variations are homothetical: with a bimarkovian matrices like $\mathbf{1}^M$ in (2), one divide by m for columns and by n for rows the absolute variations when calculating relative variations. So it is sufficient to construct only relative variations.

- For one single coefficient:

$$\sigma_{ij} = \left(z_{ij}^{*M} - z_{ij}^{M} \right)^2$$

  Remark. In this case, the real value of the term $\{i, j\}$ in the matrix $\mathbf{1}^M$ is in fact completely arbitrary and does not plays a role in the calculation (even if matrices are square). But it seems logical to consider that all terms inside are equal:

$$\mathbf{1}^M = \begin{array}{c} \left[\begin{array}{ccc} 1 & \ldots & 1 \\ .. & & .. \\ 1 & \ldots & 1 \end{array} \right] \begin{array}{c} m \\ \ldots \\ m \end{array} \\ \begin{array}{ccc} n & \ldots & n \end{array} \end{array}$$

  so, the absolute variability of an individual cell is divided by 1.

- For demanding sectors (i.e. for column vectors, σ_j):

$$\sigma_j = \frac{1}{n} \sqrt{\sum_i \left(z_{ij}^{*M} - z_{ij}^{M} \right)^2}$$

- For supplying sectors (i.e. for row vectors, σ_i):

$$\sigma_i = \frac{1}{m} \sqrt{\sum_j \left(z_{ij}^{*M} - z_{ij}^{M} \right)^2}$$

- For the whole economy, σ:

$$\sigma = \frac{1}{n\,m} \sqrt{\sum_i \sum_j \left(z_{ij}^{*M} - z_{ij}^{M} \right)^2}$$

Remark. The influence of a multiplicative parameter λ is effectively neutral:

$$\sigma_j(\lambda) = \frac{1}{n\,\lambda} \sqrt{\sum_i \left(\lambda\, z_{ij}^{*M} - \lambda\, z_{ij}^{M} \right)^2} = \sigma_j$$

$$\sigma_i(\lambda) = \frac{1}{m\,\lambda} \sqrt{\sum_j \left(\lambda\, z_{ij}^{*M} - \lambda\, z_{ij}^{M} \right)^2} = \sigma_i$$

$$\sigma(\lambda) = \frac{1}{n\,m\,\lambda} \sqrt{\sum_i \sum_j \left(\lambda\, z_{ij}^{*M} - \lambda\, z_{ij}^{M} \right)^2} = \sigma$$

so, the choice between forms (1) or (2) for matrices $\mathbf{1}^M$ is itself neutral.

C. Example

$$\mathbf{Z} = \begin{bmatrix} 5 & 5 & 6 \\ 4 & 1 & 3 \\ 3 & 4 & 5 \end{bmatrix} \begin{matrix} 16 \\ 8 \\ 12 \end{matrix} \quad \text{and } \mathbf{Z}^* = \begin{bmatrix} 2 & 3 & 8 \\ 6 & 1 & 4 \\ 1 & 2 & 6 \end{bmatrix} \begin{matrix} 13 \\ 11 \\ 9 \end{matrix}$$
$$\begin{matrix} 12 & 10 & 14 \end{matrix} \qquad\qquad \begin{matrix} 9 & 6 & 18 \end{matrix}$$

- With the ordinary biproportional filter:

$$K(\mathbf{Z},\ \mathbf{Z}^*) = \begin{bmatrix} 3.124 & 2.920 & 6.956 \\ 4.190 & 0.979 & 5.831 \\ 1.686 & 2.100 & 5.213 \end{bmatrix} \text{ and } K(\mathbf{Z}^*,\ \mathbf{Z}) = \begin{bmatrix} 4.056 & 5.228 & 6.716 \\ 5.637 & 0.807 & 1.555 \\ 2.307 & 3.964 & 5.729 \end{bmatrix}$$

For the direct projection, the results in terms of relative variabilities between $K(\mathbf{Z},\ \mathbf{Z}^*)$ and $\mathbf{Z}^*$ are respectively:

- overall: 9.63%
- for rows: 11.817%, 23.41%, 11.651%
- for columns: 24.87%, 2.17%, 12.50%.

For the reverse projection, the relative variabilities between $K(\mathbf{Z}^*,\ \mathbf{Z})$ and $\mathbf{Z}$ are:

- overall: 7.49%
- for rows: 7.54%, 27.39%, 8.39%
- for columns: 16.77%, 3.01%, 12.64%.

- For the biproportional fixed-base filter, consider the following base:

$$\mathbf{Z}^B = \begin{bmatrix} 4 & 6 & 4 \\ 3 & 2 & 5 \\ 5 & 3 & 3 \end{bmatrix} \begin{matrix} 14 \\ 10 \\ 11 \end{matrix}$$
$$\begin{matrix} 12 & 11 & 12 \end{matrix}$$

then,

$$K(\mathbf{Z},\ \mathbf{Z}^B) = \begin{bmatrix} 4.260 & 5.152 & 4.588 \\ 5.062 & 1.530 & 3.407 \\ 2.678 & 4.318 & 4.005 \end{bmatrix}$$

$$K(\mathbf{Z}^*,\ \mathbf{Z}^B) = \begin{bmatrix} 3.234 & 5.422 & 5.344 \\ 6.841 & 1.275 & 1.884 \\ 1.925 & 4.303 & 4.772 \end{bmatrix}$$

and the relative variations between $K(\mathbf{Z},\ \mathbf{Z}^B)$ and $K(\mathbf{Z}^*,\ \mathbf{Z}^B)$ are:

- overall: 8.28%
- for rows: 9.31%, 23.56%, 9.77%
- for columns: 18.23%, 3.39%, 15.54%.

And now consider,

$$\mathbf{Z}^B = \bar{\mathbf{Z}} = \frac{1}{2}(\mathbf{Z} + \mathbf{Z}^*) = \begin{bmatrix} 3.5 & 4 & 7 \\ 5 & 1 & 3.5 \\ 2 & 3 & 5.5 \end{bmatrix} \begin{matrix} 14.5 \\ 9.5 \\ 10.5 \end{matrix}$$
$$\begin{matrix} 10.5 & 8 & 16 \end{matrix}$$

then,

$$K\left(\mathbf{Z},\ \bar{\mathbf{Z}}\right) = \begin{bmatrix} 4.011 & 3.953 & 6.536 \\ 4.195 & 1.033 & 4.272 \\ 2.294 & 3.014 & 5.192 \end{bmatrix}$$

$$K\left(\mathbf{Z}^*,\ \bar{\mathbf{Z}}\right) = \begin{bmatrix} 2.925 & 4.119 & 7.456 \\ 6.008 & 0.940 & 2.552 \\ 1.567 & 2.942 & 5.991 \end{bmatrix}$$

and the relative variations between $K\left(\mathbf{Z},\ \bar{\mathbf{Z}}\right)$ and $K\left(\mathbf{Z}^*,\ \bar{\mathbf{Z}}\right)$ are:

- overall: 8.92%

  Here, $\mathbf{Z}^B$ seems a better base than $\bar{\mathbf{Z}}$ for a projection by a biproportional single-base projector because the overall relative variability is lower.

- for rows: 9.88%, 26.32%, 10.32%
- for columns: 21.29%, 2.55%, 13.17%.

- With the biproportional bimarkovian filter:

$$\mathbf{Z}^M = \begin{bmatrix} 0.853 & 1.204 & 0.943 \\ 1.468 & 0.518 & 1.014 \\ 0.679 & 1.278 & 1.043 \end{bmatrix} \begin{matrix} 3 \\ 3 \\ 3 \end{matrix} \quad \text{and } \mathbf{Z}^{*M} = \begin{bmatrix} 0.604 & 1.271 & 1.126 \\ 1.942 & 0.454 & 0.604 \\ 0.454 & 1.275 & 1.271 \end{bmatrix} \begin{matrix} 3 \\ 3 \\ 3 \end{matrix}$$
$$\begin{matrix} 3 & 3 & 3 \end{matrix} \qquad\qquad \begin{matrix} 3 & 3 & 3 \end{matrix}$$

The results in terms of relative variabilities between $\mathbf{Z}^M$ and $\mathbf{Z}^{*M}$ are:

- overall: 8.61%
- for rows: 10.54%, 21.02%, 10.67%.
- for columns: 19.37%, 3.08%, 16.80%.

D. Remarks

It is ineffective to project $K(\mathbf{Z},\ \mathbf{Z}^B)$ to $K(\mathbf{Z}^*,\ \mathbf{Z}^B)$, or $\mathbf{Z}^M$ to $\mathbf{Z}^{*M}$, by a biproportion because both matrices have the same margins (Mesnard, 1994):

$$K\left[K(\mathbf{Z},\ \mathbf{Z}^B),\quad K(\mathbf{Z}^*,\ \mathbf{Z}^B)\right] = K(\mathbf{Z},\ \mathbf{Z}^B) \text{ and } K(\mathbf{Z}^M,\ \mathbf{Z}^{*M}) = \mathbf{Z}^M$$

This is why both matrices can be compared directly.

Unlike for the biproportional filter, where a direct computation, $K(\mathbf{Z},\ \mathbf{Z}^*)$, and a reverse computation, $K(\mathbf{Z}^*,\ \mathbf{Z})$, can be done with a different result in both case, here the same results are found when $K(\mathbf{Z},\ \mathbf{Z}^B)$ (respectively $\mathbf{Z}^M$) is compared to $K(\mathbf{Z}^*,\ \mathbf{Z}^B)$ (respectively $\mathbf{Z}^{*M}$) or when $K(\mathbf{Z}^*,\ \mathbf{Z}^B)$ (respectively $\mathbf{Z}^{*M}$) is compared to $K(\mathbf{Z},\ \mathbf{Z}^B)$ (respectively $\mathbf{Z}^M$).

This is a real advantage: it is no more necessary to have a complicated and more or less empirical procedure to interpret the results by comparing two series of results. For this reason, the new method is more satisfying.

One can also remark that in both methods two biproportional projections are required, so the amount of computations remains similar, excepting that there is only one calculation of relative variabilities (with simpler computations) and no computations of absolute variabilities.

IV. Application

The bimarkovian filter will be compared to the ordinary biproportional filter for the results of an application based on data for France for the period 1980-1996. The two tables are given in the base of 1980, at the prices of 1980.

The biproportional bimarkovian filter is calculated after 30 iterations. They will be compared to similar results based on the same data but with the ordinary biproportional filter. Remember that, if the percentages of variation obtained with this method and with the bimarkovian filter can be compared (both provide relative variations), there are two ways of projection in the biproportional filter and only one in the bimarkovian filter: one will synthesize these results, direct and reverse, in a completely empirical way, by computing the mean of these two; this provide an help for the comparison with the results of the biproportional bimarkovian filter. However, the comparison must be cautious even it is done over data in relative terms.

The overall change is:

	Bimark.	Biprop. mean	Biprop. direct	Biprop. reverse	Average direct + inverse
Overall	2.20	5.30	6.61	1.23	3.92

Table 1. Comparison of methods, overall change, in %

Here, the biproportional bimarkovian filter has the lower overall relative variability.

In the following tables, the results for the biproportional bimarkovian filter will be presented in a first column, a second column will contain the results of the biproportional mean filter, a third and fourth column will present the results for the ordinary biproportional filter for direct and reverse computations are presented and a last column will indicate the average of column two an three. Table 2 presents results for column sectors and table 3 for row sectors. Higher values for relative variations are indicated in bold for the biproportional bimarkovian filter, the biproportional mean filter and for the average of the ordinary biproportional filter.

With all methods, the main result is the overwhelming role of sector T37 (*Services of Financial Institutions*), for both column and row vectors. This is caused by the strong development of exchanges between financial institutions, which can appear partially artificial because only balances are really exchanged each month. This is why in the future reform of the French national accounting system, only these balances will be taken into account. But a discussion remain concerning these phenomenons. apart this sector, the results are the following:

For other sectors, the results are the following with the biproportional bimarkovian filter:

- For column vectors, T32 (*Telecommunications and Mail*) and T06 (*Electricity, Gas and Water*) are largely changing but less than T37; also T36 (*Insurances*), T38 (*Non Marketable Services*), T22 (*Press and Publishing*), T34 (*Marketable Services to Private Individuals*), T17 (*Shipping, Aeronautics, Weapons*) are significantly changing (by more than 10%).
- For row vectors, T32 (*Telecommunications and Mail*), T15B (*Domestic Equipment Goods for Households*), T17 (*Shipping, Aeronautics, Weapons*), T24 (*Building Trade, Civil and Agricultural Engineering*), T08 (*Mining and non Ferrous Metals*), T29 (*Car Trade and Repair Services*), T35 (*Hiring, Leasing for Housing*), T22 (*Press and Publishing*).

Biproportional mean filter and the average of the ordinary biproportional filter provide very similar results. When the results of the biproportional bimarkovian filter are compared to the biproportional mean filter or to the average of the ordinary biproportional filter,

- for the list of column vectors, one must remove T38 (*Non Marketable Services*) and T22 (*Press and Publishing*), but one must add T15B (*Domestic Equipment Goods for Households*), T24 (*Building Trade, Civil and Agricultural Engineering*), T25 (*Trade*), T29 (*Car Trade and Repair Services*) and T35 (*Hiring, Leasing for Housing*).
- for the list of row vectors, one must add T12 (*Parachemestry, Pharmaceuticals*), T21 (*Paper, Cardboard*), T22 (*Press and Publishing*), T33 (*Marketable Services to Firms*), T34 (*Marketable Services to Private Individuals*). A noticeable fact is that sector T24 (*Building Trade, Civil and Agricultural Engineering*) becomes the most changing sector, a few before T37 (*Services of Financial Institutions*).

Some large differences between direct and reverse projections can be noted with the ordinary biproportional filter, for example,

- for column vectors: T10 (*Glass*), T13 (*Smelting Works, Metal Works*), T14 (*Mechanical Construction*), T15A (*Electric Professional Engineering*), T18 (*Textile Industry, Clothing Industry*), T21 (*Paper, Cardboard*), T24 (*Building Trade, Civil and Agricultural Engineering*), T25 (*Trade*), T29 (*Car Trade and Repair Services*), T34 (*Marketable Services to Private Individuals*).
- for row vectors: T21 (*Paper, Cardboard*), T22 (*Press and Publishing*), T24 (*Building Trade, Civil and Agricultural Engineering*), T29 (*Car Trade and Repair Services*), T32 (*Telecommunications and Mail*), T33 (*Marketable Services to Firms*), T34 (*Marketable Services to Private Individuals*), T35 (*Hiring, Leasing for Housing*).

These cases are indicated in italics in the following tables 2 and 3.

Sectors	Bimark.	Biprop. mean	Biprop. direct	Biprop. reverse	Average direct + inverse
T01 Farming, Forestry, Fishing	7,75	5,22	5,55	4,67	5,11
T02 Meat and Milk Products	5,73	3,94	4,53	3,6	4,07
T03 Other Agricultural and Food Products	4,99	7,05	8,57	5,34	6,96
T04 Solid Fuels	3,92	5,46	7,03	3,79	5,41
T05 Oil Products, Natural Gas	4,83	3,45	4,86	1,9	3,38
T06 Electricity, Gas and Water	**18,55**	**29,6**	28,45	30,97	**29,71**
T07 Mining and Ferrous Metals	4,23	5,74	7,49	4,24	5,87
T08 Mining and non Ferrous Metals	3,69	4,2	5,08	2,41	3,75
T09 Building Materials, Misc. Minerals	5,32	8,6	11,1	6,15	8,63
T10 Glass	3,6	9,05	***12,22***	*2,69*	7,46
T11 Basic Chemicals, Synthesized Fibers	6,48	7,78	9,42	5,05	7,23
T12 Parachemestry, Pharmaceuticals	3,95	6,92	9,05	3,91	6,48
T13 Smelting Works, Metal Works	3,76	7,48	***10,07***	*2,78*	6,43
T14 Mechanical Construction	3,04	7,41	***10,23***	*1,51*	5,87
T15A Electric Professional Engineering	5,6	8,95	***11,24***	*3,95*	7,6
T15B Domestic Equipment Goods for Households	8,48	**11,14**	11,42	11,43	**11,43**
T16 Motor Cars for Land Transport	3,08	7,7	9,52	4,44	6,98
T17 Shipping, Aeronautics, Weapons	**10,38**	**13,71**	14,53	11,56	**13,05**
T18 Textile Industry, Clothing Industry	3,24	7,27	*9,88*	*2,22*	6,05
T19 Leather and Shoe Industries	5,15	8,15	8,83	7,8	8,32
T20 Wood, Furnitures, Varied Industries	2,63	4,31	5,65	2,34	4
T21 Paper, Cardboard	2,49	8,69	***11,05***	*2,97*	7,01
T22 Press and Publishing	**13,21**	6,67	8,72	3,24	5,98
T23 Rubber, Transformation of Plastics	5,97	5,72	6,83	4,48	5,66
T24 Building Trade, Civil and Agricultural Engineering	6,52	**16,53**	***21,51***	***2,92***	**12,22**
T25 Trade	3,48	11,55	*15,28*	*3,12*	9,2
T29 Car Trade and Repair Services	6,41	**20,19**	***25,91***	***4,74***	**15,33**
T30 Hotels, Cafés, Restaurant	3,11	4,65	5,94	2,97	4,46
T31 Transports	5,23	7,76	9,68	5,03	7,36
T32 Telecommunications and Mail	**19,18**	28,71	31,49	21,73	26,61
T33 Marketable Services to Firms	2,77	4,34	5,63	2,8	4,22
T34 Marketable Services to Private Individuals	**13,18**	**15,44**	***20,07***	*4,28*	**12,18**
T35 Hiring, Leasing for Housing	3,95	**14,41**	14,32	11,91	**13,12**
T36 Insurances	**14,21**	**14,55**	19,19	9,91	**14,55**
T37 Services of Financial Institutions	**63,72**	**36,38**	29,97	49,56	**39,77**
T38 Non Marketable Services	**13,92**	7,38	7,84	6	6,92

Table 2. Comparison of methods, column vectors, in %

Sectors	Bimark.	Biprop. mean	Biprop. direct	Biprop. reverse	Average direct + inverse
T01 Farming, Forestry, Fishing	5,75	3,18	3,35	3,01	3,18
T02 Meat and Milk Products	5,75	5,61	5,61	5,69	5,65
T03 Other Agricultural and Food Products	5,86	7,4	7,01	7,94	7,48
T04 Solid Fuels	9,2	**14,5**	18,11	10	**14,06**
T05 Oil Products, Natural Gas	8,2	5,35	6,17	4,37	5,27
T06 Electricity, Gas and Water	6,54	6,21	7,46	5	6,23
T07 Mining and Ferrous Metals	3,48	3	3,38	2,94	3,16
T08 Mining and non Ferrous Metals	**11,26**	**12,15**	13,05	11,13	**12,09**
T09 Building Materials, Misc. Minerals	6,1	5,21	6,42	3,06	4,74
T10 Glass	2,82	2,73	3,08	1,78	2,43
T11 Basic Chemicals, Synthesized Fibers	2,23	2,65	2,69	2,41	2,55
T12 Parachemestry, Pharmaceuticals	6,74	9,49	9,17	11,22	**10,2**
T13 Smelting Works, Metal Works	4,05	2,53	2,86	1,83	2,35
T14 Mechanical Construction	3,71	5,44	5,94	3,93	4,94
T15A Electric Professional Engineering	7,94	5,14	5,83	3,65	4,74
T15B Domestic Equipment Goods for Households	**17,33**	**19,28**	19,43	20,48	**19,96**
T16 Motor Cars for Land Transport	3,23	2,95	3,29	1,99	2,64
T17 Shipping, Aeronautics, Weapons	**14,78**	**20,82**	21,21	18,51	**19,86**
T18 Textile Industry, Clothing Industry	3,62	2	2,4	1,62	2,01
T19 Leather and Shoe Industries	3,44	4,47	4,02	5,27	4,65
T20 Wood, Furnitures, Varied Industries	4,88	5,51	7,45	3,11	5,28
T21 Paper, Cardboard	3,53	**14,29**	***17,62***	***3,05***	**10,34**
T22 Press and Publishing	9,24	**12,66**	***15,32***	***3***	9,16
T23 Rubber, Transformation of Plastics	4,93	3,96	4,06	3,88	3,97
T24 Building Trade, Civil and Agricultural Engineering	**14,62**	**37,23**	***42,26***	***8,1***	**25,18**
T29 Car Trade and Repair Services	**10,69**	**32,14**	***38,45***	***6,96***	**22,71**
T30 Hotels, Cafés, Restaurant	2,6	3,12	4,16	2,15	3,16
T31 Transports	3,43	4,13	5,24	2,3	3,77
T32 Telecommunications and Mail	**20,25**	**32,21**	***37,44***	***8,35***	**22,9**
T33 Marketable Services to Firms	6,86	**13,64**	***17,14***	***3,37***	**10,26**
T34 Marketable Services to Private Individuals	4,13	**15,79**	***19,12***	***5,8***	12,46
T35 Hiring, Leasing for Housing	**10,3**	**23,37**	***27,61***	***8,7***	**18,16**
T36 Insurances	6	6,96	6,95	7,19	7,07
T37 Services of Financial Institutions	**57,65**	**35,04**	*28,22*	*68,79*	**48,51**

Table 3. Comparison of methods, row vectors, in %

This can be summarized in a two series of figures, where two columns are reported. When points are along the first diagonal, both methods provide the same result. When points are to the right of the diagonal, as T37 in figure 4, the variation found with the biproportional bimarkovian filter is higher than with the biproportional mean filter, and conversely when points are above the diagonal as T06 or T32 in the same figure.

As it can be seen in tables 2 and 3, more than half the number of sectors are above the diagonal: often the biproportional bimarkovian filter provide lower estimation for relative variabilities than the ordinary biproportional filter.

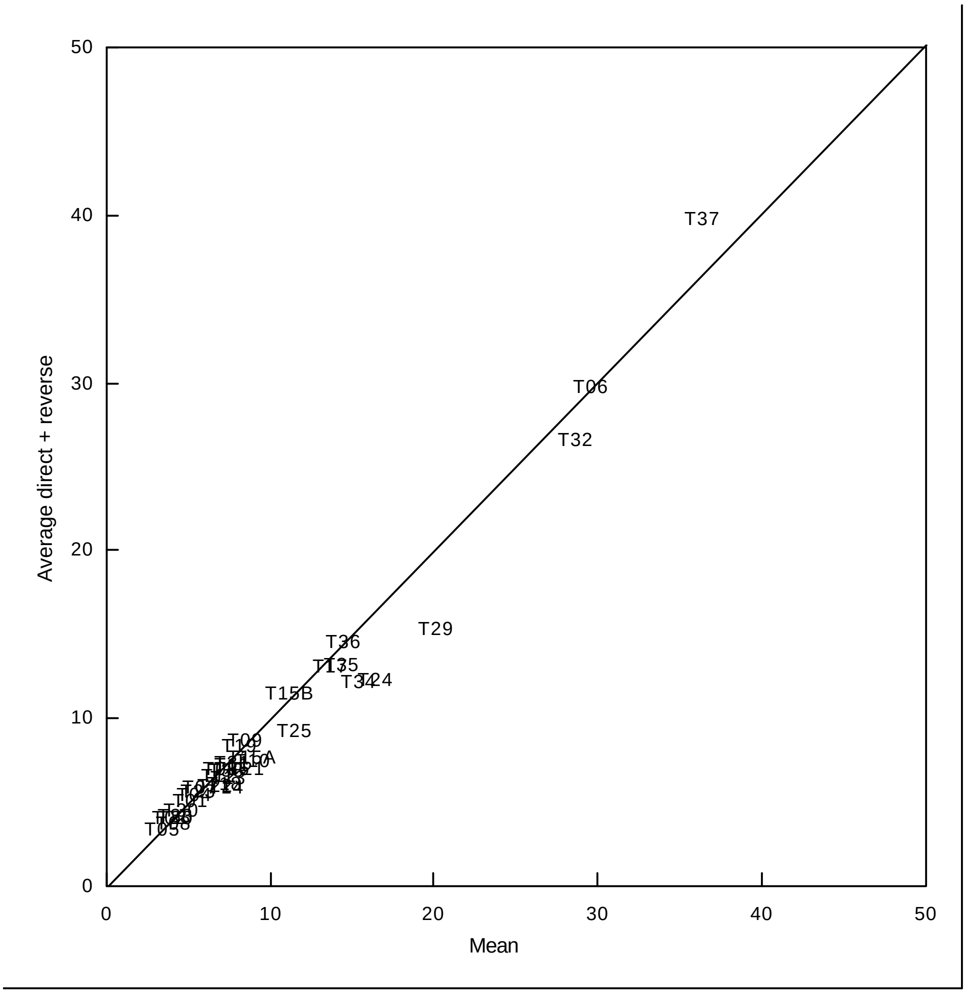

Figure 3. Comparison of methods: mean Vs average direct + reverse, for columns

In figure 3, when the biproportional mean filter is compared to the ordinary biproportional filter for columns, points seem to be correctly aligned along the first diagonal: ordinary biproportional filter and biproportional mean filter provide similar results.

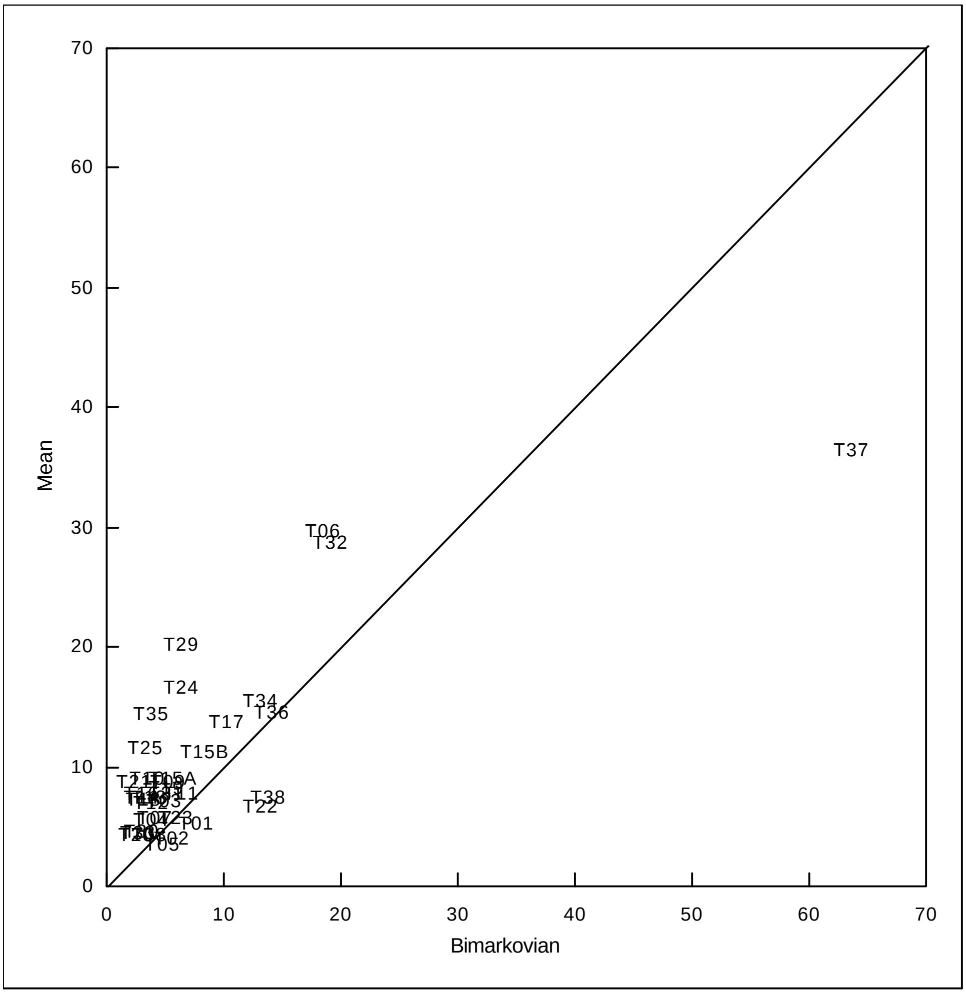

Figure 4. Comparison of methods: bimarkovian Vs mean, for columns

In figure 4, when comparing the biproportional bimarkovian filter to the biproportional mean filter for columns, except for point T37 (*Services of Financial Institutions*) which is to the right of the first diagonal, many points are to the left. Note that, except for T37, the sectors which are far from the diagonal are not generally the sectors with the greater size in $\bar{\mathbf{Z}}$, so the differences between both methods are not linked to a simple size effect: if only large sectors would have be far from the diagonal, the biproportional bimarkovian filter would have detect only large sectors and it would have be trivial.

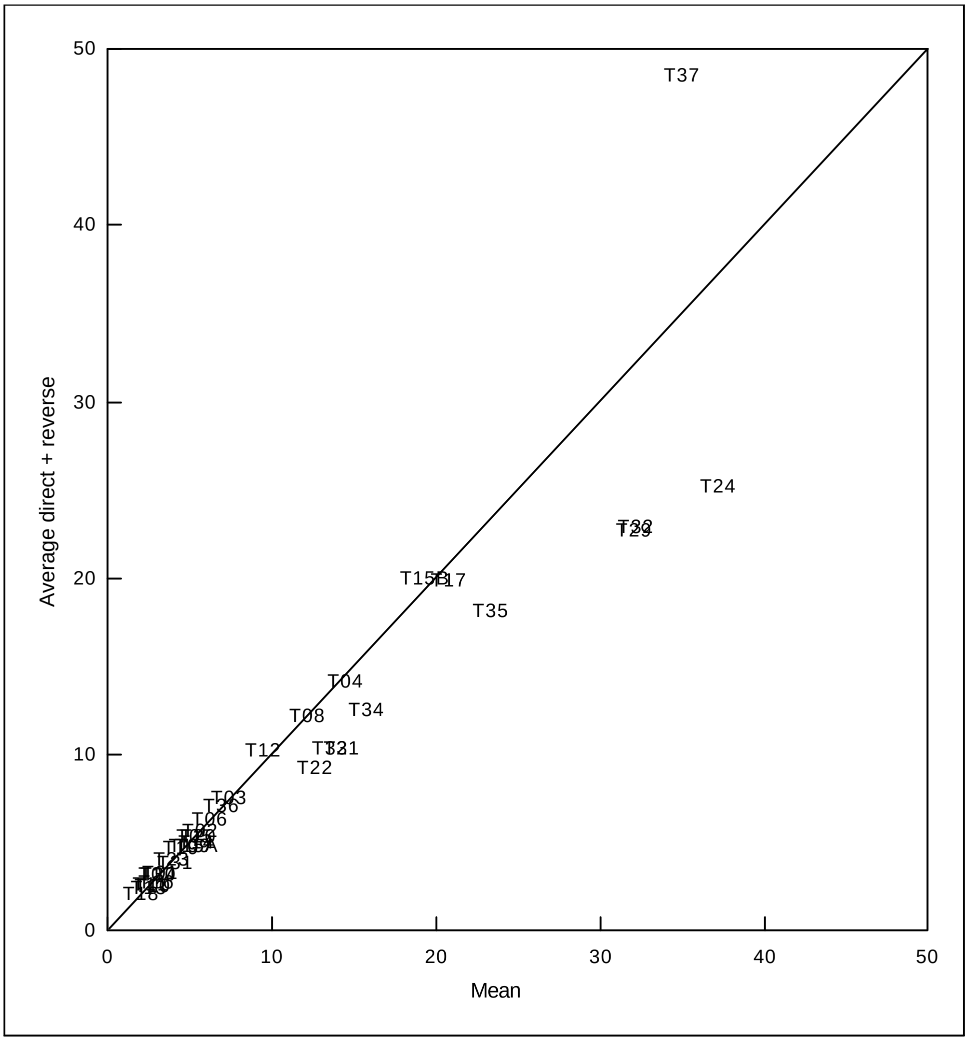

Figure 5. Comparison of methods: mean Vs average direct + reverse, for rows

In figure 5 when comparing the biproportional mean filter to the ordinary biproportional filter for rows, some points are to the right of the first diagonal, for example T24 (*Building Trade, Civil and Agricultural Engineering*), T29 (*Car Trade and Repair Services*) and T32 (*Telecommunications and Mail*), T35 (*Hiring, Leasing for Housing*), but T37 (*Services of Financial Institutions*) is far to the left.

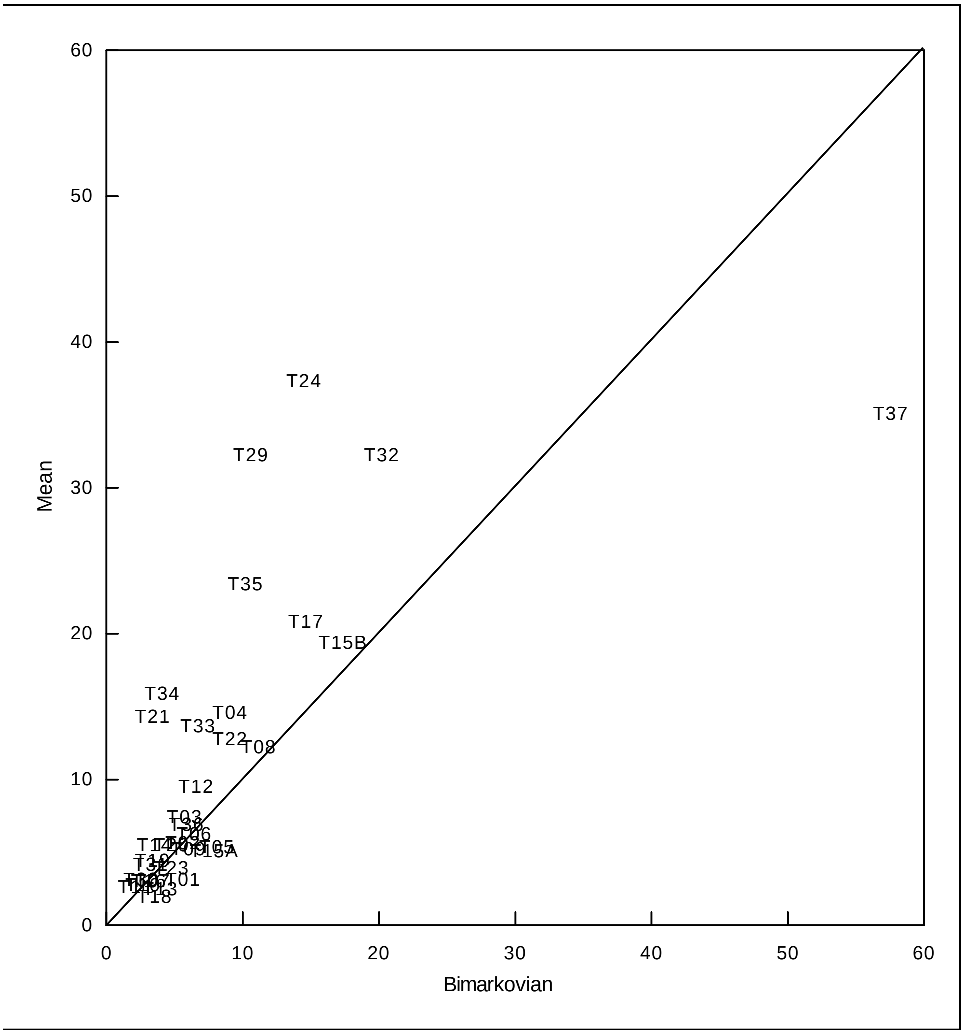

Figure 6. Comparison of methods: bimarkovian Vs mean, for rows

In figure 6 when the biproportional bimarkovian filter is compared to the biproportional mean filter, almost all points are to the right of the first diagonal for high values as T32 (*Telecommunications and Mail*), T24 (*Building Trade, Civil and Agricultural Engineering*), T29 (*Car Trade and Repair Services*), T35 (*Hiring, Leasing for Housing*), etc., except for T37 (*Services of Financial Institutions*). Again, except for T37 and T24, sectors which are far from the diagonal are not the larger.

V. Conclusion

In the ordinary biproportional filter, one matrix **Z** (respectively $\mathbf{Z}^*$) is projected on the margins of the another matrix $\mathbf{Z}^*$ (respectively **Z**) by the mean of biproportion and then the projected matrix is compared to $\mathbf{Z}^*$ (respectively **Z**); this method avoids to remove the differential effects of sectors without predetermining if the model is demand driven or supply driven, but there are two different results possibly strongly diverging. A first generalization of this method, the *biproportional mean filter*, which avoids this difficulty, provides results which are close to the ordinary biproportional filter. However, the differences of sizes of sectors are not removed. In the biproportional bimarkovian filter, both matrices **Z** and $\mathbf{Z}^*$ are transformed into bimarkovian matrices by the mean of biproportion and then these two transformed matrices are compared. This method also avoids to predetermine if the model is demand or supply driven by removing the effect of differential growth of sectors and with only a single result but with the advantage that the differences of sizes of sectors are removed: so it is more satisfying. Results are not exactly the same with other biproportional filter, but the heavier tendencies are conserved in the application for years 1980-1996 in France.

VI. Annex

A. *Remind about the computation of a biproportion*

The result of a biproportion, $K(\mathbf{Z}, \mathbf{Z}^*)$, is equal to **U Z V**, where **U** and **V** are diagonal matrices, of which terms guarantee that $K(\mathbf{Z}, \mathbf{Z}^*)$ have the same row and column margins than $\mathbf{Z}^*$; for example, the following algorithm is correct for this purpose (Bachem and Korte, 1979):

$$u_i = \frac{m_{i\bullet}}{\sum_{j=1}^{m} v_j\, z_{ij}} \text{ , for all } i \text{ , and } v_j = \frac{m_{\bullet j}}{\sum_{i=1}^{n} u_i\, z_{ij}} \text{, for all } j \tag{5}$$

As many algorithms are possible to obtain the same result, it is demonstrated in (Mesnard, 1994) that all algorithms K lead to the same results [4] [5]. Among these algorithms, considering three matrices, **Z** a known matrix to be projected to the margins of another matrix **M** (i.e. under constraints of margins: $\sum_i y_{ij} = \sum_i m_{ij}$ and $\sum_j y_{ij} = \sum_j m_{ij}$) and **Y** the searched matrix result of the projection, one have:

- the algorithm (1),
- Stone's empirical method RAS,

[4] One must understand "the same theoretical result", because there can be differences in terms of speed of computations and in terms of effects of successive rounds. Among these algorithms, there is Stone's RAS method and the concept of biproportion was first formalized by Bacharach (1970).

[5] Balinski and Demange (1989) have studied the axioms of biproportion in real numbers and in integers (see also (Aït-Sahalia, Balinski and Demange, 1988)); this is applied to voting problems; see also Balinski and Young (1994) and Balinski and Gonzalez (1996).

- the maximization of entropy (Wilson, 1970):

$$\min \sum_i \sum_j y_{ij} \log y_{ij}$$

under the constraint $\sum_i \sum_j y_{ij} c_{ij}$ where **C** is a cost matrix linked to **Z**,

- Kullback's minimization of information [6] (Kullback, 1959):

$$\min \sum_i \sum_j y_{ij} \log \frac{y_{ij}}{z_{ij}}$$

So the computation of a biproportion is a safe operation. However, a confusion must no be done: the unicity of biproportion concerns the choice of an algorithm and not the fact that the solution of one specific algorithm leading to a biproportional form is unique.

B. Other methods to compute a biproportion

Also, there exists other methods to found a matrix **Y** projection of one matrix **Z** to the margins of one another matrix **M** (i.e. under constraints of margins: $\sum_i y_{ij} = \sum_i m_{ij}$ and $\sum_j y_{ij} = \sum_j m_{ij}$):

- the minimization of the quadratic deviation (Frobenius norm of the difference matrix):

$$\min \sum_i \sum_j (y_{ij} - z_{ij})^2$$

- the minimization of the absolute differences:

$$\min \sum_i \sum_j |y_{ij} - z_{ij}|$$

- or as a generalization of two preceding, the minimization of the Hölder norm at the power p:

$$\min \sum_i \sum_j |y_{ij} - z_{ij}|^p$$

knowing that the Hölder norm (Rotella and Borne, p. 78) is:

$$\|\mathbf{Y} - \mathbf{Z}\|_p = \left[\sum_i \sum_j |y_{ij} - z_{ij}|^p \right]^{1/p}$$

- Pearson's χ^2:

$$\sum_i \sum_j \frac{(y_{ij} - z_{ij})^2}{z_{ij}}$$

- Neyman's χ^2:

$$\sum_i \sum_j \frac{(y_{ij} - z_{ij})^2}{y_{ij}}$$

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[6] For example, Aït-Sahalia, Balinski and Demange (1988) establish that the matrix that minimizes the information criteria is unique (note that in their paper, they make a confusion between the maximization of entropy and Kullback's minimization of information).

But generally these methods lead to various problems, like non-linearities or non-differentiabilities in the found system as for Neyman or absolute differences, or negative terms in **Y** as for the minimization of the Frobenius norm. Negative terms are impossible to explain in an input-output context: if **Z** have no negative terms, how justify in an economic view point, the existence of some negative terms inside the projected matrix **Y**?

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